The canonical structure of Poincaré gauge theory

M. Blagojević — Divčibare, 12–14 September 2011

Contents

1. Introduction

A. Localization of Poincaré symmetry (Utiyama 1956; Kibble, Sciama 1961)

- **Rigid** Poincaré group = the isometry group of M_4 (translations and Lorentz rotations)
Invariance of the matter field Lagrangian:

$$\delta_0\mathcal{L}_M(\phi,\partial_k\phi)=0 \quad \text{where} \quad \delta_0\phi := (\varepsilon^k P_k + \tfrac{1}{2}\omega^{ij}M_{ij})\phi$$

- **Localization**: $\varepsilon^k \to \varepsilon^k(x)$, $\omega^{ij} \to \omega^{ij}(x)$ $\Rightarrow$ invariance of $\mathcal{L}_M(\phi,\partial_k\phi)$ lost!
Invariance restored with the help of compensating fields $b^k = b^k{}_\mu dx^\mu$, $A^{ij} = A^{ij}{}_\mu dx^\mu$:

$$\mathcal{L}_M(\phi, d\phi) \to b\mathcal{L}_M(\phi, \nabla\phi)$$

where $b=\det(b^k{}_\mu)$ and $\nabla = d + \frac{1}{2}A^{ij}\Sigma_{ij}$ is the covariant derivative.
Field strengths – the torsion and the curvature:

$$T^i = \nabla b^i,\ \ R^{ij} = dA^{ij} + A^i{}_k A^{kj}.$$

Gauge field Lagrangian:

$$\mathcal{L}_G = R + T^2 + R^2$$

- **Geometric** interpretation: Riemann-Cartan geometry of spacetime, with $\nabla g_{ij} = 0$

B. Canonical formalism with constraints (Dirac, Bargmann and others 1950s)

- Gauge invariance of $\mathcal{L} \Rightarrow$ some dyn. variables are unphysical (arbitrary funct's of time)
in electrodynamics: $F_{00} = 0 \Rightarrow A_0$ is dynamically undetermined

- What happens in the canonical formalism?

(a) there are constraints in the phase space;
in electrodynamics: $\pi^0 = \partial\mathcal{L}/\partial\partial_0 A_0 \approx 0$ (primary constraint)

(b) for consistency, constraints must be preserved in time:
$\dot{\pi}^0 = \{\pi^0, H_T\} = \partial_\alpha \pi^\alpha \approx 0$ (secondary constraint) $\quad \alpha = 1,2,3$

(c) Hamiltonian contains arbitrary multipliers:
$H_T = -A_0(\partial_\alpha \pi^\alpha) + u\pi^0 + \cdots$, where $\partial_\alpha \pi^\alpha$ and π^0 are first class (FC) constraints

(d) FC constraints produce gauge transformations:
$\delta_0 A_\mu = \{A_\mu, G\} = \partial_\mu \varepsilon$, where G is the gauge generator

2. Hamiltonian and constraints

In **PG theory**, independent dyn. variables are: tetrad b^i, Lorentz connection $A^{ij} = -A^{ji}$

- **Einstein-Cartan** theory: $\mathcal{L} = -aR +$ matter (Nester 1977, Nikolić 1995)

Field equations: $R_{i\mu} - \frac{1}{2} b_{i\mu} R \sim$ energy-momentum current $\tau_{i\mu}$
torsion $\sim$ spin current $\sigma_{ij\mu}$

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- **Quadratic PG** th.: $\mathcal{L} \sim R + R^2 + T^2 +$ matter (parity preserving, 9 free param's)

Canonical analysis (MB and Nikolić 1983; Nikolić 1984):
$b^i{}_0$ and $A^{ij}{}_0$ are unphysical variables $\Rightarrow$ Dirac-ADM form of the Hamiltonian:

$$\mathcal{H}_c = b^i{}_0\mathcal{H}_i - \tfrac{1}{2}A^{ij}{}_0\mathcal{H}_{ij} \qquad \mathcal{H}_T = \mathcal{H}_c + u \cdot \phi$$

$\mathcal{H}_i$ and $\mathcal{H}_{ij}$ are always present – sure constraints
If the parameters in Lagrangian take on some critical values $\Rightarrow$ additional, if-constraints ϕ

Results:
– All critical parameters and if-constraints are **identified**
– Consistency analysis of the constraints is **completed**

3. Conserved charges

Dirac believed that all FC constraints generate unphysical (gauge) transformations
Castellani (1981) formulated an algorithm for construction of the canonical gauge generator
$\Rightarrow$ the canonical gauge generator G for PG theory (MB, Nikolić and Vasilić 1988)

- **Boundary conditions $\Rightarrow$ asymptotic symmetries $\Rightarrow$ conservation laws**

Technical procedure (Regge and Teitelboim 1974):
The gauge generator $G = G[\varphi, \partial\varphi; \pi, \partial\phi]$ acts on the fields via the PB operation
$\Rightarrow$ it must be differentiable – achieved by adding a surface term: $\tilde{G} = G + E$
$\Rightarrow$ the conserved charge $=$ the value of $\tilde{G} \approx E$ (since $G \approx 0$)

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- In PG theory (MB and Vasilić 1988)

– assume that spacetime is asymptotically Minkowskian
– construct $\tilde{G} = (\tilde{P}_i, \tilde{M}_{ij})$ to describe the asymptotic Poincaré symmetry
– identify the conserved charges:

energy-momentum $=$ the value of $\tilde{P}_i \approx E_i$
angular momentum $=$ the value of $\tilde{M}_{ij} \approx E_{ij}$

4. Nonlinear constraint effects

PG theory defined by 9 parameters in $\mathcal{L}$ – what is an acceptable choice?

- Some of the **standard theoretical requirements:**

– massive propagating modes with $v < c$ (no tachions) and $E > 0$ (no ghosts)
(Sezgin and Nieuwenhuizen 1980)

– well posed initial value problem
there is a strong correlation with the values of can. critical parameters (Dimakis 1989)

– classical solutions should have positive energy (Katanaev 1987)

1st and 2nd requirement based on the linear approximation scheme
3rd - of limited practical value

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- A **new, canonical test** of viability of PG (Chen, Nester and Yo 1991)

In PG theory, linearization with respect to a given background may change the number and/or type of constraints $\Rightarrow$ modified physical content

New requirement:

the canonical structure of the nonlinear theory should **remain the same** after linearization

In **PG theory:**

there are only two good torsion modes, scalar ($J^P = 0^+$) and pseudoscalar ($J^P = 0^-$), defined with respect to an M_4 background
$\Rightarrow$ they uniquely define acceptable PG Lagrangians

Torsion = dark energy!? (Shie, Nester and Yo 2008)
Homogeneous and isotropic cosmology contains only $J^P = 0^\pm$ torsion modes.
Lagrangian with scalar torsion mode:

$$\mathcal{L} \sim a_0 R + b_0 R^2 + a_1(\ T^2\text{-terms}\ )$$

spin 0^+ torsion mode is able to produce **dark energy effects**:

– the present-day accelerated expansion of the Universe, which slows down
– a realistic value of the Hubble constant

5. Covariant Hamiltonian formalism

The standard canonical formalism is based on the concept of **physical state at a time** t
$\Rightarrow$ **not manifestly** covariant
Dirac (1949) considered dynamical evolution along **any** direction in spacetime

- **Manifestly covariant canonical formalism** (Nester 1991, Hecht 1993, and others), based on

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \quad \rightarrow \quad \pi^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}$$

In PG theory:

– construct the first order form of $\mathcal{L}$: $\mathcal{L} = T^i \wedge \pi_i + R^{ij} \wedge \pi_{ij} - V(\tau, \rho, b)$
– define $\mathcal{H}(\xi)$, ξ timelike; then, allow ξ to be either timelike or spacelike, like Dirac
– $H[\xi] = \int_\mathcal{M} \mathcal{H}(\xi)$ is not differentiable
$\tilde{H}[\xi] = H[\xi] + E[\xi]$, where $E[\xi] = \int_\Sigma B(\xi)$ is the **conserved charge**

Nester et al. found a **universal** expression for $E(\xi)$, valid for a **large number** of b. c., such as: M_4, (A)dS, and Bondi (radiation) b. c.

A very practical result!

6. 3D gravity with torsion

- **Topological 3D gravity with torsion** (Mielke and Baekler 1991)

$$\mathcal{L} = -aR - 2\Lambda_0 + \alpha_3 \mathcal{L}_{CS}(A) + \alpha_4 \varepsilon^{ijk} T_{ijk}$$

– BTZ black hole with torsion (Garcıa et al. 2003)
– AdS asymptotic symmetry ⇒ two Virasoro algebras with **classical** central charges
Classical central charges produce **quantum** effects, such as
black hole entropy (MB and Cvetković 2006):

$$S = \frac{2\pi r_+}{4G} + 4\pi^2 \alpha_3 \Big(p r_+ - 2\frac{r_-}{\ell} \Big) \quad \text{where } p = \text{ the strength of torsion}$$

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- **3D gravity with propagating torsion** (MB and Cvetković, work in progress):

$$\mathcal{L} = -aR - 2\Lambda_0 + L_{T^2} + L_{R^2}$$

Exploring the **canonical stability under linearization** with respect to
– **Minkowski and AdS** background.
One expects to see certain **differences!**

Concluding remarks

- **Constrained Hamiltonian formalism** clarifies basic dynamical features of PG theory:

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– general expression for the conserved charges
– criterion for good PG theories
– simple explanation of the mysterious concept of dark energy

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- Classical canonical structure has had an important influence on the foundation of both the canonical and the path-integral method of quantization