Extra gauge symmetries in BHT gravity

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Gravity: new ideas for unsolved problems
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Outline

The talk is based on papers:

- M. Blagojević and B. Cvetković, Extra gauge symmetries in BHT gravity, JHEP **03** (2011) 139;
- M. Blagojević and B. Cvetković, Hamiltonian analysis of BHT massive gravity, JHEP **01** (2011) 082.

- Recently, Bergshoeff, Hohm and Townsend (BHT) proposed a parity conserving theory of gravity in three dimensions (3D), defined by adding certain curvature-squared terms to the Einstein-Hilbert action.
- With the cosmological constant Λ_0, $\sigma = \pm 1$ and $a = 1/16\pi G$ the action takes the form:

$$I = a \int d^3x \sqrt{g} \left(\sigma R - 2\Lambda_0 + \frac{1}{m^2} K \right) , \; K := R_{ij} R^{ij} - \frac{3}{8} R^2 \, .$$

  where R_{ij} is the Ricci tensor and R the scalar curvature.
- When BHT gravity is linearized around the Minkowski ground state, it is found to be equivalent to the Fierz-Pauli theory for a free massive spin-2 field. The theory is ghosts-free, unitary and renormalizable.

- The overall picture is changed when we go over to the (A)dS background, where various dynamical properties, such as unitarity, gauge invariance or boundary behavior, become more complex.
- The particle content of the BHT gravity depends on the values of coupling constants. Maximally symmetric vacuum state defined by $G_{ij} = \Lambda_{\rm eff}\, \eta_{ij}$, where $\Lambda_{\rm eff}$ the effective cosmological constant, solves the BHT field equations if $\Lambda_{\rm eff}$ solves a simple quadratic equation:

$$\Lambda_{\rm eff}^2 + 4m^2\sigma\Lambda_{\rm eff} - 4m^2\Lambda_0 = 0\,.$$

- For $\Lambda_0/m^2 = -1$, two solutions for $\Lambda_{\rm eff}$ coincide, and we have a unique vacuum state. In that case, one finds an extra *gauge symmetry* in the linear approximation, and massive modes become partially *massless*.

- Dynamical characteristics of a gravitational theory take a particularly clear form in the constrained Hamiltonian approach.
- Analyzing the nature of constraints in the fully *nonlinear* BHT gravity, we discovered the special role of an extra condition:

$$\Omega^{00} := \sigma g^{00} + \frac{G^{00}}{2m^2} \neq 0\,.$$

  which, when applied to a maximally symmetric solution, takes the familiar form $\Lambda_0/m^2 \neq -1$.
- In the region of space where $\Omega^{00} \neq 0$, the resulting theory is found to possess *two* Lagrangian degrees of freedom, which corresponds to two helicity states of the massive graviton excitation.

- We extend our investigation to the *critical point* $\Lambda_0/m^2 = -1$ in the maximally symmetric sector of the theory, by studying the canonical structure of the BHT gravity *linearized* around the maximally symmetric background.
- Analyzing the constraint structure of the linearized theory, we construct the canonical generator of *extra gauge symmetry*, which is responsible for transforming two massive graviton excitations into a single, partially massless mode.
- By comparing these results with those obtained nonperturbatively, we can understand how the canonical structure of the BHT gravity is changed in the linearization process.

Notations and conventions

- Our conventions are as follows:
  - the Latin indices $(i, j, k, ...)$ refer to the local Lorentz frame, the Greek indices $(\mu, \nu, \lambda, ...)$ refer to the coordinate frame, and both run over 0,1,2;
  - the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor ε^{ijk} is normalized to $\varepsilon^{012} = 1$.
- Our notation follows the Poincaré gauge theory (PGT) framework in 3D:
  - fundamental dynamical variables are the triad field b^i and the Lorentz connection ω^i (1-forms),
  - $T^i = db^i + \varepsilon^{ijk}\omega_j b_k$ and $R^i = d\omega^i + \frac{1}{2}\varepsilon^i{}_{jk}\omega^j\omega^k$ are the corresponding field strengths, the torsion and the curvature (2-forms),
  - the relation to the standard 4D notation is given by $\omega^{ij} = -\varepsilon^{ij}{}_k\omega^k$, $R^{ij} = -\varepsilon^{ij}{}_k R^k$.

- The BHT massive gravity with a cosmological constant is formulated as a gravitational theory in Riemannian spacetime. Instead of using the standard Riemannian formalism, we find it more convenient to use the triad field and the spin connection as fundamental dynamical variables.
- The description of the BHT massive gravity can be technically simplified as follows.
  (a) We use the triad field b^i and the spin connection ω^i as independent dynamical variables.
  (b) The Riemannian nature of the connection is ensured by imposing the vanishing of torsion with the help of the Lagrange multiplier $\lambda^i = \lambda^i{}_\mu dx^\mu$.
  (c) Finally, by introducing an auxiliary field $f^i = f^i{}_\mu dx^\mu$, we transform the term K into an expression linear in curvature.

- The Lagrangian is given by:

$$L = a\left(2\sigma b^i R_i - \frac{1}{3}\Lambda_0 \varepsilon_{ijk} b^i b^j b^k + \frac{1}{m^2} L_K\right) + \lambda^i T_i\,,$$

$$L_K = R_i f^i - V_K\,, \quad V_K := \frac{1}{4} f_i{}^\star\left(f^i - f\,b^i\right) = \mathcal{V}_K\,\hat\epsilon\,,$$

where $f = f^k{}_k$ and $\hat\epsilon = b^0 b^1 b^2$ is the volume 3-form.

- Variation with respect to basic dynamical variables yields field equations, which imply that spacetime is Riemannian and:

$$f_{ij} = 2L_{ij}\,, \qquad \lambda_{ij} = \frac{2a}{m^2} C_{ij}\,,$$

where L_{ij} and C_{ij} are the Schouten and the Cotton tensor:

$$L_{ij} = R_{ij} - \frac{1}{4}\eta_{ij} R\,, \qquad C_{ij} = \varepsilon_i{}^{mn}\nabla_m L_{nj}\,.$$

- Basic field equation of BHT gravity takes the form:

$$\sigma G_{ij} - \Lambda_0 \eta_{ij} - \frac{1}{2m^2} K_{ij} = 0\,,$$

where $K_{ij} := \mathcal{T}_{ij} - 2(\nabla_m C_{in})\varepsilon^{mn}{}_j$ and $\mathcal{T}_{ij}$ is the energy-momentum tensor associated to $\mathcal{L}_K$.

- An important set of algebraic consequences of the field equations reads

$$\begin{aligned} &f_{\mu\nu} = f_{\nu\mu}\,, \\ &\lambda_{\mu\nu} = \lambda_{\nu\mu}\,, \qquad \lambda = 0\,, \\ &\sigma f + 3\Lambda_0 + \frac{1}{2m^2}\mathcal{V}_K = 0\,. \end{aligned}$$

- The last consequence represents trace of the basic field equation.

- Introducing the notation $Q_A = (b^i{}_\mu, \omega^i{}_\mu, f^i{}_\mu, \lambda^i{}_\mu)$, we now consider the linearized form of the theory around a maximally symmetric solution $\bar{Q}_A$, characterized by

$$\bar{G}_{ij} = \Lambda_{\text{eff}}\, \eta_{ij}\,, \qquad \bar{f}^i{}_\mu = -\Lambda_{\text{eff}}\, \bar{b}^i{}_\mu\,, \qquad \bar{\lambda}^i{}_\mu = 0\,,$$

- The linearization is based on the expansion

$$Q_A = \bar{Q}_A + \widetilde{Q}_A\,,$$

where $\widetilde{Q}_A$ is a small excitation around $\bar{Q}_A$.

- The trace of the basic (linearized) field equation reads:

$$\left(\sigma + \frac{\Lambda_{\text{eff}}}{2m^2}\right) \bar{h}_i{}^\mu \left(\tilde{f}^i{}_\mu + \Lambda_{\text{eff}}\, \tilde{b}^i{}_\mu\right) = 0\,.$$

- For the *critical value* of parameters, $\Lambda_{\text{eff}} + 2\sigma m^2 = 0$ ($\Lambda_0/m^2 = -1$), this equation is identically satisfied.

- When we have a maximally symmetric background, the critical condition implies that the massive graviton of the linearized BHT gravity (with two helicity states) becomes a (single) partially massless mode; simultaneously, there appears an extra gauge symmetry in the theory.
- By a systematic analysis of the related canonical structure, we discover that this symmetry has the following form:

$$\begin{aligned}
\delta_E \tilde{b}^i{}_\mu &= \epsilon \bar{b}^i{}_\mu \,, \\
\delta_E \tilde{\omega}^i{}_\mu &= -\varepsilon^{ijk} \bar{b}_{j\mu} \bar{h}_k{}^\nu \bar{\nabla}_\nu \epsilon \,, \\
\delta_E \tilde{f}^i{}_\mu &= -2\bar{\nabla}_\mu (\bar{h}^{i\nu} \bar{\nabla}_\nu \epsilon) + \Lambda_{\text{eff}}\, \epsilon \bar{b}^i{}_\mu \,, \\
\delta_E \tilde{\lambda}^i{}_\mu &= 0 \,,
\end{aligned}$$

where ϵ is an infinitesimal gauge parameter.

- The primary constraints are given by:

$$\phi_i{}^0 := \pi_i{}^0 \approx 0\,, \qquad \phi_i{}^\alpha := \pi_i{}^\alpha - \varepsilon^{0\alpha\beta}\lambda_{i\beta} \approx 0\,,$$

$$\Phi_i{}^0 := \Pi_i{}^0 \approx 0\,, \quad \Phi_i{}^\alpha := \Pi_i{}^\alpha - 2a\varepsilon^{0\alpha\beta}\left(\sigma b_{i\beta} + \frac{1}{2m^2} f_{i\beta}\right) \approx 0$$

$$p_i{}^\mu \approx 0\,, \qquad\qquad P_i{}^\mu \approx 0\,.$$

- The canonical Hamiltonian can be conveniently written as:

$$\mathcal{H}_c = b^i{}_0\mathcal{H}_i + \omega^i{}_0\mathcal{K}_i + f^i{}_0\mathcal{R}_i + \lambda^i{}_0\mathcal{T}_i + \frac{a}{m^2} b\mathcal{V}_K\,,$$

$$\mathcal{H}_i = -\varepsilon^{0\alpha\beta}\left(a\sigma R_{i\alpha\beta} - a\Lambda_0\varepsilon_{ijk} b^j{}_\alpha b^k{}_\beta + \nabla_\alpha\lambda_{i\beta}\right)\,,$$

$$\mathcal{K}_i = -\varepsilon^{0\alpha\beta}\left(a\sigma T_{i\alpha\beta} + \frac{a}{m^2}\nabla_\alpha f_{i\beta} + \varepsilon_{ijk} b^j{}_\alpha\lambda^k{}_\beta\right)\,,$$

$$\mathcal{R}_i = -\frac{a}{2m^2}\varepsilon^{0\alpha\beta} R_{i\alpha\beta}\,, \qquad \mathcal{T}_i = -\frac{1}{2}\varepsilon^{0\alpha\beta} T_{i\alpha\beta}\,.$$

- Going over to the total Hamiltonian,

$$\mathcal{H}_T = \mathcal{H}_c + u^i{}_\mu \phi_i{}^\mu + v^i{}_\mu \Phi_i{}^\mu + w^i{}_\mu p_i{}^\mu + z^i{}_\mu P_i{}^\mu \,,$$

we find that the consistency conditions of the primary constraints $\pi_i{}^0$, $\Pi_i{}^0$, $p_i{}^0$ and $P_i{}^0$ yield the secondary ones:

$$\hat{\mathcal{H}}_i := \mathcal{H}_i + \frac{a}{m^2} b \mathcal{T}_i^0 \approx 0 \,, \quad \mathcal{K}_i \approx 0 \,,$$

$$\hat{\mathcal{R}}_i := \mathcal{R}_i + \frac{a}{2m^2} b (f_i{}^0 - f h_i{}^0) \approx 0 \,, \quad \mathcal{T}_i \approx 0 \,.$$

- Tertiary constraints read:

$$\theta_{\mu\nu} := f_{\mu\nu} - f_{\nu\mu} \,, \quad \psi_{\mu\nu} := \lambda_{\mu\nu} - \lambda_{\nu\mu} \,,$$

while quartic are given by:

$$\chi := \lambda \approx 0 \,, \quad \varphi := \sigma f + 3\Lambda_0 + \frac{1}{2m^2} \mathcal{V}_K \approx 0 \,.$$

- The consistency condition for the quartic constraint φ has the form:

$$\{\varphi, H_T\}_1^* = \Omega^{\mu\nu} z'_{\mu\nu} \approx 0\,,$$

$$\Omega^{\mu\nu} := \sigma g^{\mu\nu} + \frac{1}{4m^2}\left(f^{\mu\nu} - f g^{\mu\nu}\right)\,,$$

where ${z^i}_0{}' := {z^i}_0 - {f^i}_k {u^k}_0$

- This relation determines the multiplier z'_{00}, provided the coefficient Ω^{00} does not vanish.
- The total Hamiltonian can be expressed in terms of the first class constraints (up to an ignorable square of constraints) as follows:

$$\hat{\mathcal{H}}_T = {b^i}_0 \bar{\mathcal{H}}_i + {\omega^i}_0 \bar{\mathcal{K}}_i + {u^i}_0 {\pi_i}^{0\prime\prime} + {v^i}_0 {\Pi_i}^0\,,$$

where ${\pi_i}^{0\prime\prime} = {\pi_i}^0 + \lambda_{ji} p^{j0} + f_{ji} P^{j0}\,.$

- The complete classification of constraints in the reduced space R_1, defined by second class constraints $(\phi_i{}^\alpha, \Phi_i{}^\alpha, p_i{}^\alpha, P_i{}^\alpha)$ is:

	First class	Second class
Primary	$\pi_i{}^{0\prime\prime}, \Pi_i{}^0$	$p_i{}^0, P_i{}^0$
Secondary	$\bar{\mathcal{H}}_i, \bar{\mathcal{K}}_i$	$\mathcal{T}_i, \hat{\mathcal{R}}'_i$
Tertiary		$\theta_{0\beta}, \theta_{\alpha\beta}, \psi_{0\beta}, \psi_{\alpha\beta}$
Quartic		χ, φ

- The number of independent dynamical degrees of freedom in R_1 is $N^* = 4$ and the theory exhibits 2 local Lagrangian degree of freedom.

- At the critical point complete classification of constraints in R_1 is given by:

	First class	Second class
Primary	$\tilde{\pi}_i{}^{0\prime\prime}, \tilde{\Pi}_i{}^0, \widetilde{P}^{00}$	$\tilde{p}_i{}^0, \widetilde{P}^{\alpha 0}$
Secondary	$\hat{\mathcal{H}}'_i, \mathcal{K}_i, \hat{\mathcal{R}}^{00}$	$\mathcal{T}_i, \hat{\mathcal{R}}^{\alpha\prime}$
Tertiary	$\psi_{\alpha\beta}$	$\theta_{0\alpha}, \theta_{\alpha\beta}, \psi_{0\alpha}$
Quartic		χ

where $\hat{\mathcal{H}}'_i = \hat{\mathcal{H}}_i - \Lambda_{\text{eff}}\hat{\mathcal{R}}_i$, $\hat{\mathcal{R}}^{00} = \bar{h}_i{}^0\hat{\mathcal{R}}^i - \bar{h}_i{}^0\bar{\nabla}_\beta(\bar{b}^i{}_0\widetilde{P}^{\beta 0})$.

- We obtain that the number of physical modes in the phase space is $N^* = 2$, and consequently, the BHT theory at the critical point exhibits one Lagrangian degree of freedom.

- The presence of an extra primary FC constraint $\widetilde{P}^{00}$ implies the existence of an extra gauge symmetry. Its canonical construction is simplifies in the reduced phase space R_2:

$$R_2: \qquad \theta_{\beta 0} \equiv \tilde{f}_{\beta 0} - \tilde{f}_{0\beta} = 0\,, \qquad \widetilde{P}^{\beta 0} = 0\,,$$

  to eliminate the variables $\tilde{f}_{\beta 0}$ and $\widetilde{P}^{\beta 0}$.

- Castellani's algorithm leads to the following canonical generator in R_2:

$$\begin{aligned} G_E &= -2\ddot{\epsilon}\widetilde{P}^{00} + \dot{\epsilon}\left[-2\hat{\mathcal{R}}^{0\prime} + 2(\bar{h}^{i0}\bar{\nabla}_0\bar{b}^i{}_0)\widetilde{P}^{00} + \varepsilon_{ijk}\bar{h}^{i0}\bar{b}^j{}_0\widetilde{\Pi}^k{}_0\right] \\ &\quad +\epsilon\Big[\varepsilon^{0\alpha\beta}\bar{b}^i{}_\alpha\tilde{\lambda}_{i\beta} + \tilde{\pi}_0{}^0 - \varepsilon_{ijk}\bar{\nabla}_\alpha(\bar{h}^{i\alpha}\bar{b}^j{}_0\widetilde{\Pi}^k{}_0) + 2\bar{\nabla}_\alpha\hat{\mathcal{R}}^{\alpha\prime} \\ &\quad -2\bar{\nabla}_\alpha(\bar{h}_i{}^\alpha\widetilde{P}^{00}\bar{\nabla}_0\bar{b}^i{}_0) + \Lambda_{\text{eff}}\,\bar{g}_{00}\widetilde{P}^{00}\Big]\,. \end{aligned}$$

- The canonical structure of the BHT gravity at the critical point $\Lambda_0/m^2 = -1$ does not remain the same after linearization.
- We are led to conclude that the canonical consistency of the BHT gravity, expressed by the stability of its canonical structure under linearization, is violated at the critical point $\Lambda_0/m^2 = -1$.
- It is interesting to examine whether the partially massless modes exist within $R + T^2 + R^2$ Poincaré gauge theory.
- Preliminary results for spin 0^+ mode show that at the critical point theory is canonically consistent.