Open string in the weakly curved background

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Outline

- action describing the propagation, equations of motion, boundary condition
- weakly curved background
- canonical method
- bounary conditions as Dirac constraints
- infinite number of constraints
- gathering into σ-dependent constraints
- looking for the compact form of the σ-dependent constraints
- solving constraints
- effective theory on the solution
- non-commutativity of the space-time coordinates

Action

- open string propagation

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g}\Big[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}[x] + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}[x]\Big] \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta}, (\varepsilon^{01} = -1)$$

  - conformal gauge $\quad g_{\alpha\beta} = e^{2F} \eta_{\alpha\beta}$
- minimal action principle
  - equation of motion $\quad \ddot{x}^\mu = x''^\mu - 2B^\mu{}_{\nu\rho} \dot{x}^\nu x'^\rho$

    $$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$
  - boundary condition

    $$\gamma^0_\mu \Big|_{\sigma=0,\pi} \equiv \frac{\delta \mathcal{L}}{\delta x'^\mu}\Big|_{\sigma=0,\pi} = \Big[G_{\mu\nu} x'^\nu - 2B_{\mu\nu}\dot{x}^\nu\Big]\Big|_{\sigma=0,\pi} = 0$$

Weakly curved background

- consistency of the theory implies conformal invariance on the quantum level
- space-time equations of motion

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0$$

$$D_\rho B^\rho{}_{\mu\nu} = 0$$

- solution

$$G_{\mu\nu}[x] = const$$

$$B_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho$$

$$b_{\mu\nu} = const, \quad B_{\mu\nu\rho} = const$$

- we choose infinitesimal $B_{\mu\nu\rho}$ and we work up to the terms linear in it

Boundary conditions as constraints

- Dirac constraints conserved in time $\gamma_\mu^n \equiv \dot{\gamma}_\mu^{n-1}, \quad (n \geq 1)$
- the explicit form of the constraints on the equations of motion

$$\gamma_\mu^{2n} = \gamma_\mu^{(2n)} - \frac{2}{3} \sum_{k=0}^{n-1} \alpha_{2n}^k (Q_\mu^k)^{(2n-2k-1)} + 4 b_\mu{}^\nu \sum_{k=0}^{n-1} \alpha_{2n}^k (R_\nu^k)^{(2n-2k-2)}$$

$$\gamma_\mu^{2n+1} = \tilde{\gamma}_\mu^{(2n+1)} - \frac{2}{3} \sum_{k=0}^{n-1} \alpha_{2n}^k (R_\mu^k)^{(2n-2k-1)} + 4 b_\mu{}^\nu \sum_{k=0}^{n} \alpha_{2n+2}^k (Q_\nu^k)^{(2n-2k)}$$

- auxiliary functions

$$\gamma_\mu = G_{\mu\nu} x'^\nu - 2B_{\mu\nu}\dot{x}^\nu, \quad \tilde{\gamma}_\mu = G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu} x'^\nu,$$

$$Q_\mu^n = B_{\mu\nu\rho}\dot{x}^{(n)\nu} x^{(n+1)\rho}, \quad R_\mu^n = B_{\mu\nu\rho}\left[x^{(n+2)\nu} x^{(n+1)\rho} + \dot{x}^{(n)\nu}\dot{x}^{(n+1)\rho}\right]$$

$$\alpha_{2n}^k = (-4)^k \binom{n}{k+1}, \quad k = 0, \cdots, n-1$$

σ dependent constraints

- we treat constraints with even and odd indices separately

$$\Gamma_\mu^S(\sigma) \equiv \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} \gamma_\mu^{2n}\Big|_{\sigma=0} = 0, \quad \Gamma_\mu^A(\sigma) \equiv \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} \gamma_\mu^{2n+1}\Big|_{\sigma=0} = 0$$

- keeping in mind the explicit form of the constraints, the σ-dependent constraints will have the form

$$\begin{aligned} \Gamma_\mu^S(\sigma) &= \gamma_\mu^S(\sigma) - \frac{2}{3}\Gamma_\mu^Q(\sigma) + 4b_\mu{}^\nu \tilde{\Gamma}_\nu^R(\sigma) \\ \Gamma_\mu^A(\sigma) &= \tilde{\gamma}_\mu^A(\sigma) - \frac{2}{3}\Gamma_\mu^R(\sigma) + 4b_\mu{}^\nu \tilde{\Gamma}_\nu^Q(\sigma) \end{aligned}$$

$$\gamma_\mu^S(\sigma) \equiv \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} \gamma_\mu^{(2n)}\Big|_{\sigma=0}, \qquad \tilde{\gamma}_\mu^A(\sigma) \equiv \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} \gamma_\mu^{(2n+1)}\Big|_{\sigma=0}$$

σ dependent constraints

$$\Gamma_\mu^Q = \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\frac{\sigma^{2n}}{(2n)!}\alpha_{2n}^k(Q_\mu^k)^{(2n-2k-1)}\Big|_{\sigma=0},$$

$$\Gamma_\mu^R = \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\frac{\sigma^{2n+1}}{(2n+1)!}\alpha_{2n}^k(R_\mu^k)^{(2n-2k-1)}\Big|_{\sigma=0},$$

$$\tilde{\Gamma}_\mu^R = \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\frac{\sigma^{2n}}{(2n)!}\alpha_{2n}^k(R_\mu^k)^{(2n-2k-2)}\Big|_{\sigma=0},$$

$$\tilde{\Gamma}_\mu^Q = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{\sigma^{2n+1}}{(2n+1)!}\alpha_{2n+2}^k(Q_\mu^k)^{(2n-2k)}\Big|_{\sigma=0}$$

[0] note that we have inverted the order of summation

σ dependent constraints

- after nontrivial calculations the explicit expressions for these four functions was obtained

$$
\begin{aligned}
\Gamma^{Q}_{\mu}(\sigma) &= \frac{1}{2} B_{\mu\nu\rho}\Big[\dot{Q}^{\nu} q'^{\rho} + \frac{1}{2}\dot{\bar{q}}^{\nu}\bar{q}^{\rho}\Big] \\
\Gamma^{R}_{\mu}(\sigma) &= \frac{1}{2} B_{\mu\nu\rho}\int_0^{\sigma} d\eta \Big[\bar{q}''^{\nu}\bar{q}^{\rho} + \dot{Q}^{\nu}\dot{q}'^{\rho}\Big] \\
\tilde{\Gamma}^{R}_{\mu}(\sigma) &= B_{\mu\nu\rho}\Big[\frac{1}{2} q'^{\nu}\bar{q}^{\rho} + \frac{1}{2}\dot{Q}^{\nu}\dot{\bar{q}}^{\rho}\Big] \\
\tilde{\Gamma}^{Q}_{\mu}(\sigma) &= B_{\mu\nu\rho}\frac{\partial}{\partial\sigma}\Big[\dot{Q}^{\nu}\bar{q}^{\rho}\Big], \qquad Q^{\mu}(\sigma) = \int_0^{\sigma} d\eta\, q^{\mu}(\eta)
\end{aligned}
$$

- where we separated even and odd coordinate parts

$$
x^{\mu}(\sigma) = q^{\mu}(\sigma) + \bar{q}^{\mu}(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu}\Big|_0 + \sum_{n=0}^{\infty}\frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu}\Big|_0
$$

Compact form of the constraints

$$\begin{aligned}\Gamma^S_\mu(\sigma) &= G_{\mu\nu}\bar{q}'^\nu - 2b_{\mu\nu}\dot{q}^\nu - \frac{2}{3}B_{\mu\nu\rho}\Big[\dot{q}^\nu q^\rho + \frac{1}{2}\dot{Q}^\nu q'^\rho + \frac{3}{2}\dot{\bar{q}}^\nu\bar{q}^\rho\Big] \\ &+ 2b_\mu{}^\nu B_{\nu\rho\upsilon}\Big[q'^\rho\bar{q}^\upsilon + \dot{Q}^\rho\dot{\bar{q}}^\upsilon\Big] \\ \Gamma^A_\mu(\sigma) &= G_{\mu\nu}\dot{\bar{q}}^\nu - 2b_{\mu\nu}q'^\nu - \frac{2}{3}B_{\mu\nu\rho}\Big[q'^\nu q^\rho + \frac{1}{2}\dot{Q}^\nu\dot{q}^\rho + \frac{3}{2}\bar{q}'^\nu\bar{q}^\rho\Big] \\ &+ 2b_\mu{}^\nu B_{\nu\rho\upsilon}\frac{\partial}{\partial\sigma}\Big[\dot{Q}^\rho\bar{q}^\upsilon\Big]\end{aligned}$$

Compact canonical form of the constraints

$$
\begin{aligned}
\Gamma^S_\mu(\sigma) &= G^E_{\mu\nu}[q]\bar{q}'^\nu - \frac{2}{\kappa} B_\mu{}^\nu[q] p_\nu \\
&+ \Big[- 2(bh + hb) + 6h[bq] + 24bh[bq]b \Big]'_{\mu\nu} \bar{q}^\nu \\
&- \frac{1}{\kappa} \Big[h - 12bh[bq] \Big]'^{\ \nu}_{\mu} P_\nu \\
&- \frac{3}{\kappa} \Big[\bar{h} + 4bh[b\bar{q}] \Big]_\mu^{\ \nu} \bar{p}_\nu + \frac{6}{\kappa^2} \Big[bh[\bar{p}] \Big]_\mu^{\ \nu} P_\nu
\end{aligned}
$$

$$
\begin{aligned}
G^E_{\mu\nu}(G, B) &\equiv G_{\mu\nu} - 4B_{\mu\rho}(G^{-1})^{\rho\sigma} B_{\sigma\nu} \\
h_{\mu\nu}[x] &\equiv \frac{1}{3} B_{\mu\nu\rho} x^\rho, \quad h_{\mu\nu} \equiv h_{\mu\nu}[q], \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu}[\bar{q}] \\
P_\mu(\sigma) &= \int_0^\sigma d\eta p_\mu(\eta)
\end{aligned}
$$

Compact canonical form of the constraints

$$
\begin{aligned}
\Gamma^A_\mu(\sigma) &= \frac{1}{\kappa}\bar{p}_\mu \\
&+ \Big[-\bar{h} + 12 b\bar{h}b + 4h[b\bar{q}]b - 12bh[b\bar{q}] \Big]_{\mu\nu} \bar{q}'^\nu \\
&+ \frac{2}{\kappa}\Big[3b\bar{h} + h[b\bar{q}] \Big]_\mu{}^\nu p_\nu \\
&+ \frac{2}{\kappa}\Big[3b\bar{h} - h[b\bar{q}] \Big]_\mu'{}^\nu P_\nu - \frac{1}{\kappa^2} h_\mu{}^\nu[p] P_\nu
\end{aligned}
$$

$$
\begin{aligned}
h_{\mu\nu}[x] &\equiv \frac{1}{3} B_{\mu\nu\rho} x^\rho, \quad \bar{h}_{\mu\nu} = h_{\mu\nu}[\bar{q}] \\
P_\mu(\sigma) &= \int_0^\sigma d\eta\, p_\mu(\eta)
\end{aligned}
$$

Solving constraints iteratively

- zeroth order in $B_{\mu\nu\rho}$

$$\bar{q}_0'^{\mu} = -\theta_0^{\mu\nu} p_\nu, \qquad \theta_0^{\mu\nu} \equiv -\frac{2}{\kappa}(g^{-1})^{\mu\rho} b_{\rho\sigma} (G^{-1})^{\sigma\nu}$$

$$\bar{p}_\mu^0 = 0$$

- symmetric constraint on the zeroth order solution

$$\Gamma_\mu^S(\sigma) = G_{\mu\nu}^E[q]\left[\bar{q}'^{\nu} + \theta^{\nu\rho}[q] p_\rho + \frac{1}{2}\Lambda_-'^{\nu\rho}[q] P_\rho\right]$$

$$G_{\mu\nu}^E(G,B) \equiv G_{\mu\nu} - 4B_{\mu\rho}(G^{-1})^{\rho\sigma} B_{\sigma\nu}$$

$$\theta^{\mu\nu}(G,B) \equiv -\frac{2}{\kappa}\left[G_E^{-1}(G,B)\right]^{\mu\rho} B_{\rho\sigma}(G^{-1})^{\sigma\nu}$$

$$\Lambda_\pm^{\mu\nu}[q] \equiv \theta^{\mu\nu}[q] \mp \frac{6}{\kappa}(G_E^{-1})^{\mu\nu}[bq]$$

Solving constraints iteratively

- antisymmetric constraint on the zeroth order solution

$$\begin{aligned}\Gamma^A_\mu(\sigma) &= \frac{1}{\kappa}\bar{p}_\mu + \Big[- h[\theta_0 P]\theta_0 - \frac{6}{\kappa} bh[\theta_0 P]g^{-1} \\ &+ \frac{1}{\kappa^2} h[g^{-1}P]g^{-1} + \frac{6}{\kappa} bh[g^{-1}P]\theta_0 \Big]_\mu^{\ \nu} p_\nu\end{aligned}$$

$$\begin{aligned}g_{\mu\nu} &\equiv G_{\mu\nu} - 4b_{\mu\rho}(G^{-1})^{\rho\sigma} b_{\sigma\nu} \\ h_{\mu\nu}[x] &\equiv \frac{1}{3} B_{\mu\nu\rho} x^\rho\end{aligned}$$

Constrained space-time coordinates and momenta

- on the solution of constraints $\Gamma^S_\mu(\sigma)=0$ and $\Gamma^A_\mu(\sigma)=0$, up to the terms linear in $B_{\mu\nu\rho}$ we have

$$
\begin{aligned}
x^\mu(\sigma) &= q^\mu(\sigma)+\bar{q}^\mu(\sigma) \\
&= q^\mu(\sigma)-\int_0^\sigma d\eta\Big[\theta^{\mu\nu}[q(\eta)]p_\nu(\eta)+\frac{1}{2}\Lambda'^{\mu\nu}_{-}[q(\eta)]P_\nu(\eta)\Big] \\
\pi_\mu(\sigma) &= p_\mu(\sigma)+\bar{p}_\mu(\sigma) \\
&= p_\mu+\Big[Gb^{-1}\beta[\bar{q}_0]g^{-1}\Big]_\mu^{\ \nu}p_\nu
\end{aligned}
$$

$$
\beta_{\mu\nu}[\bar{q}_0]=2\Big[bh[\bar{q}_0]b-3b^2h[\bar{q}_0]-\frac{1}{4}bh[b^{-1}\bar{q}_0]+3b^2h[b^{-1}\bar{q}_0]b\Big]_{\mu\nu}
$$

The effective theory in terms of the effective variables

$$
\begin{aligned}
S^{eff} &= \int d\tau \int_{-\pi}^{\pi} d\sigma [\pi_\mu \dot{x}^\mu - \mathcal{H}_c(x,\pi)]\Big|_{\Gamma_\mu = 0} \\
&= \int_{\Sigma_1} d\xi \left[p_\mu \dot{q}^\mu - \mathcal{H}_c^{eff}(q,p) \right] \\
\mathcal{H}_c^{eff}(q,p) &= \frac{1}{2\kappa} p_\mu (G_E^{-1})^{\mu\nu}[q]\; p_\nu + \frac{\kappa}{2} q'^\mu G^E_{\mu\nu}[q]\; q'^\nu \\
&- 2q'^\mu \left[(\bar{h} + 4b\bar{h}b)g^{-1} \right]_\mu^{\ \nu} p_\nu
\end{aligned}
$$

The effective theory (and consequently the effective Hamiltonian $\mathcal{H}_c^{eff}(q,p)$), should depend on the effective variables q^μ, p_μ in exactly the same way as the original theory (the original Hamiltonian $\mathcal{H}_c(x,\pi)$) depends on original variables x^μ, π_μ.

[0] Σ_1 means the integration over σ in the interval $[-\pi, \pi]$

Initial versus effective theory

- looking at Hamiltonians
  - the canonical variable transition

$$x^\mu, \pi_\mu \to q^\mu, p_\mu$$

  - the background transition

$$\begin{aligned} G_{\mu\nu} &\to G^{E}_{\mu\nu}[q] \equiv G^{eff}_{\mu\nu}[q] \\ B_{\mu\nu}[x] &\to (\bar{h} + 4b\bar{h}b)_{\mu\nu} = -\frac{\kappa}{2}\Big[g(\theta^{\mu\nu}[-\theta_0 P] - \theta_0^{\mu\nu})g\Big]_{\mu\nu} \\ &\equiv B^{eff}_{\mu\nu}[-\theta_0 P] \end{aligned}$$

- looking at Lagrangians
  - effective Lagrangian

$$\mathcal{L}^{eff}(q) = \frac{\kappa}{2}\dot{q}^\mu G^{E}_{\mu\nu}[q]\dot{q}^\nu - \frac{\kappa}{2}q'^\mu G^{E}_{\mu\nu}[q]q'^\nu + 2\kappa q'^\mu B^{eff}_{\mu\nu}[2b\dot{Q}]\dot{q}^\nu$$

  - transition

$$x^\mu \to q^\mu, \quad G_{\mu\nu} \to G^{eff}_{\mu\nu}[q], \quad B_{\mu\nu}[x] \to B^{eff}_{\mu\nu}[2b\dot{Q}]$$

Effective Kalb-Ramond field

- the effective Kalb-Ramond field depends on the Ω-odd variable $B^{eff}_{\mu\nu}[\bar{q}_0]$
- we consider only the zeroth order value $\bar{q}_0^\mu$, because $\bar{q}^\mu$ appears only as an argument of the infinitesimally small field $B^{eff}_{\mu\nu}$
- $\bar{q}_0^\mu = -2(G^{-1}b)^\mu{}_\nu \dot{Q}^\nu$ in the Lagrangian approach
- the zeroth order equation of motion $\partial_+\partial_- q^\mu = 0$
- the solution $q^\mu(\sigma) = f^\mu(\sigma^+) + f^\mu(\sigma^-)$, $(\sigma^\pm = \tau \pm \sigma)$
- $\dot{q}^\mu(\sigma) = f'^\mu(\sigma^+) - f'^\mu(\sigma^-)$
- $\dot{Q}^\mu(\sigma) = f^\mu(\sigma^+) - f^\mu(\sigma^-) = \tilde{q}^\mu(\sigma)$
- $\tilde{q}^\mu(\sigma)$ is T-dual coordinate

Effective theory once again

- on the solution of the constraints the zeroth order space-time coordinate can be rewritten as $x^\mu = q^\mu + 2(G^{-1}b)^\mu{}_\nu \tilde{q}^\nu$
- the effective metric depends on the first term q^μ and the effective Kalb-Ramond field on the second term $2b\tilde{q}$
- we can formally rewrite both the effective Hamiltonian and the effective action, as if the effective background fields depend on the same argument

$$
\begin{aligned}
H_c^{eff} &= \int_{-\pi}^{\pi} d\sigma \Big[\frac{1}{2\kappa} p_\mu (G_{eff}^{-1})^{\mu\nu}[x] p_\nu + \frac{\kappa}{2} q'^\mu G^{eff}_{\mu\nu}[x] q'^\nu \\
&\quad -2q'^\mu B^{eff}_{\mu\rho}[x] (G^{-1})^{\rho\nu} p_\nu \Big] \\
S^{eff} &= \kappa \int d\tau \int_{-\pi}^{\pi} d\sigma \Big[\frac{1}{2} \eta^{\alpha\beta} G^{eff}_{\mu\nu}[x] + \epsilon^{\alpha\beta} B^{eff}_{\mu\nu}[x] \Big] \partial_\alpha q^\mu \partial_\beta q^\nu
\end{aligned}
$$

Non-commutativity relation for the space-time coordinates

$$
\begin{aligned}
{}^{\star}\{x^{\mu}(\sigma), x^{\nu}(\bar{\sigma})\} &= 2\Big[E^{\mu\nu}(\bar{\sigma}) - E^{\nu\mu}(\sigma)\Big]\theta(\sigma+\bar{\sigma}) \\
&- 2\Big[I^{\mu\nu}(\bar{\sigma}) + I^{\nu\mu}(\sigma)\Big]\theta(\sigma-\bar{\sigma}) \\
E^{\mu\nu}[q,P] &= \frac{1}{2}\Lambda_{+}^{\mu\nu}[q] - I^{\mu\nu}[P] \\
&= \frac{1}{2}\theta^{\mu\nu}[q] - \frac{3}{\kappa}(G_{eff}^{-1})^{\mu\nu}[bq] - I^{\mu\nu}[P] \\
I^{\mu\nu}[P] &= \frac{1}{2}\theta^{\mu\nu\rho}P_{\rho} - \frac{1}{2}\theta_{eff}^{\mu\nu}[-\theta_0 P] + \frac{1}{4}\theta_0^{\nu\alpha}\partial_{\alpha}\Lambda_{-}^{\mu\rho}P_{\rho}
\end{aligned}
$$

- the term with $\theta(\sigma-\bar{\sigma})$ obviously does not contribute on the boundary
- only the antisymmetric part of $E^{\mu\nu}$ contributes on the boundary

Removing the center of mass variable

$$x^\mu(\sigma) = X^\mu(\sigma) + x^\mu_{cm}, \qquad x^\mu_{cm} = \frac{1}{\pi}\int_0^\pi d\sigma x^\mu(\sigma)$$

- the commutation relation on the string end-points

$$^\star\{X^\mu(0), X^\nu(0)\} = -\theta^{\mu\nu}[q(0)] - \frac{4}{\pi} I^{[\mu\nu]}[_0 P_{cm}]$$

$$^\star\{X^\mu(\pi), X^\nu(\pi)\} = \theta^{\mu\nu}[q(\pi)] + \frac{4}{\pi} I^{[\mu\nu]}[^0 P_{cm}]$$

$$\begin{aligned} {}_\circ P^{cm}_\mu &= \frac{1}{\pi}\int_0^\pi d\sigma \int_0^\sigma d\eta P_\mu(\eta), \\ {}^\circ P^{cm}_\mu &= \frac{1}{\pi}\int_0^\pi d\sigma \int_\sigma^\pi d\eta P_\mu(\eta) \end{aligned}$$

Non-commutativity relation on the boundary

- the commutation relation on the string end-points explicitly

$$
\begin{aligned}
{}^{\star}\{X^\mu(0), X^\nu(0)\} = & -\theta^{\mu\nu}[q(0)] - \frac{3}{2\pi}\theta^{\mu\nu\rho}{}_0 P^{cm}_\rho + \frac{3}{2\pi}\theta^{\mu\nu}_{eff}[-\theta_0\ {}_0 P^{cm}] \\
& - \frac{3}{\pi\kappa}\left[\theta_0^{\nu\rho}\partial_\rho (G_E^{-1})^{\mu\sigma} - \theta_0^{\mu\rho}\partial_\rho (G_E^{-1})^{\nu\sigma}\right]{}_0 P^{cm}_\sigma
\end{aligned}
$$

$$
\begin{aligned}
{}^{\star}\{X^\mu(\pi), X^\nu(\pi)\} = & \ \theta^{\mu\nu}[q(\pi)] + \frac{3}{2\pi}\theta^{\mu\nu\rho 0} P^{cm}_\rho - \frac{3}{2\pi}\theta^{\mu\nu}_{eff}[-\theta_0{}^0 P^{cm}] \\
& + \frac{3}{\pi\kappa}\left[\theta_0^{\nu\rho}\partial_\rho (G_E^{-1})^{\mu\sigma} - \theta_0^{\mu\rho}\partial_\rho (G_E^{-1})^{\nu\sigma}\right]{}^0 P^{cm}_\sigma
\end{aligned}
$$

- the term $\theta^{\mu\nu}[q]$ is standard one
- the part of the second term has been obtained but only on the boundary
- other terms are our improvement

Non-commutativity relation on the string interior

$$^{\star}\{X^\mu(\sigma), X^\nu(\bar\sigma)\} = 2l^{[\mu\nu]}[P^a(\sigma,\bar\sigma)] + 2l^{(\mu\nu)}[P^s(\sigma,\bar\sigma)]$$

$$\begin{aligned} P^a_\mu(\sigma,\sigma) &= 2P_\mu(\sigma)\left[\frac{1}{2} - \frac{\sigma}{\pi}\right] + \frac{2}{\pi}\left[{}_0P_\mu(\sigma) - {}_0P^{cm}_\mu\right] \\ P^s_\mu(\sigma,\sigma) &= 0, \qquad\qquad \sigma = \bar\sigma \neq 0,\pi \end{aligned}$$

$$\begin{aligned} P^a_\mu(\sigma,\bar\sigma) &= P_\mu(\sigma) - \frac{1}{\pi}\left[\bar\sigma P_\mu(\bar\sigma) + \sigma P_\mu(\sigma)\right] \\ &+ \frac{1}{\pi}\left[{}_0P_\mu(\sigma) + {}_0P_\mu(\bar\sigma)\right] - \frac{2}{\pi}\,{}_0P^{cm}_\mu \\ P^s_\mu(\sigma,\bar\sigma) &= -P_\mu(\sigma) - \frac{1}{\pi}\left[\bar\sigma P_\mu(\bar\sigma) - \sigma P_\mu(\sigma)\right] \\ &+ \frac{1}{\pi}\left[{}_0P_\mu(\sigma) - {}_0P_\mu(\bar\sigma)\right], \qquad\qquad \sigma > \bar\sigma \end{aligned}$$

Non-commutativity relation on the string interior

$$
\begin{aligned}
P^a_\mu(\sigma,\bar\sigma) &= P_\mu(\bar\sigma) - \frac{1}{\pi}\Big[\bar\sigma P_\mu(\bar\sigma) + \sigma P_\mu(\sigma)\Big] \\
&+ \frac{1}{\pi}\Big[{}_0P_\mu(\sigma) + {}_0P_\mu(\bar\sigma)\Big] - \frac{2}{\pi}\,{}_0P^{cm}_\mu \\
P^s_\mu(\sigma,\bar\sigma) &= P_\mu(\bar\sigma) - \frac{1}{\pi}\Big[\bar\sigma P_\mu(\bar\sigma) - \sigma P_\mu(\sigma)\Big] \\
&+ \frac{1}{\pi}\Big[{}_0P_\mu(\sigma) - {}_0P_\mu(\bar\sigma)\Big], \qquad\qquad \bar\sigma > \sigma
\end{aligned}
$$

Canonical quantization

- the straightforward generalization of the standard quantization procedure is not possible
- x^μ and π_ν do not commute, but neither do the x^μ's themselves
- two possible rules of associating the operators
- first possibility
  - we could consider the operator functions of coordinates and momenta $\hat{f}(\hat{x}, \hat{\pi})$ and normal ordering among both x's and x and π

Canonical quantization

- second possibility
  - consider the effective variables q^μ and p_μ as fundamental variables
  - introduce some normal ordering :: for corresponding operators $\hat{q}^\mu$ and $\hat{p}_\mu$
  - which defines $\hat{x}^\mu$ and is not needed for $\hat{\pi}_\mu$

$$\begin{aligned}
\hat{x}^\mu(\sigma) &= \hat{q}^\mu(\sigma) - \int_0^\sigma d\eta :\Big[\theta^{\mu\nu}[\hat{q}(\eta)]\hat{p}_\nu(\eta) + \frac{1}{2}\Lambda_-^{\prime\mu\nu}[\hat{q}(\eta)]\hat{P}_\nu(\eta)\Big]: \\
\hat{\pi}_\mu(\sigma) &= 2\hat{p}_\mu(\sigma) + \Big[Gb^{-1}\beta[-\theta_0\hat{P}]g^{-1}\Big]_\mu^{\ \nu}\hat{p}_\nu
\end{aligned}$$

  - now assign the operator $:\hat{f}(\hat{q},\hat{p}):$ to any function $f(x,\pi)$ by

$$\begin{aligned}
f(x,\pi) \quad \to \quad & :\hat{f}\Big\{\hat{q}^\mu - \int_0^\sigma d\sigma_0\Big[\theta^{\mu\nu}[\hat{q}]\hat{p}_\nu + \frac{1}{2}\Lambda_-^{\prime\mu\nu}[\hat{q}]\hat{P}_\nu\Big], \\
& 2\hat{p}_\mu + \Big[Gb^{-1}\beta[-\theta_0\hat{P}]g^{-1}\Big]_\mu^{\ \nu}\hat{p}_\nu\Big\}:
\end{aligned}$$

References

- Ljubica Davidović, Branislav Sazdović
  Non-commutativity in weakly curved background by canonical methods
  Apr 2010. (Received Mar 15, 2011) Published in Phys.Rev.D83:066014,2011.
- Ljubica Davidović, Branislav Sazdović
  Non-commutativity parameters depend not only on the effective coordinate but on its T-dual as well
  Jun 2011. Published in JHEP 1108:112,2011.