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Is general relativity a gauge theory of the translation group in disguise?

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This contribution is dedicated to Milutin Blagojević on the occasion of his 67th birthday in appreciation of his contributions to theoretical physivcs and of his friendship.

A forthcomming Reader with Commentaries “On Gauge Theories of Gravity” by Milutin and myself has been freely used. All references can be found there. *file Milutin6702.tex*

Contents

Up to here, I use plausibility considerations such that everybody could get an idea of the subjct. Subsequently, I will become a bit more technical:

1. General structure of the Poincaré gauge theory (PG)

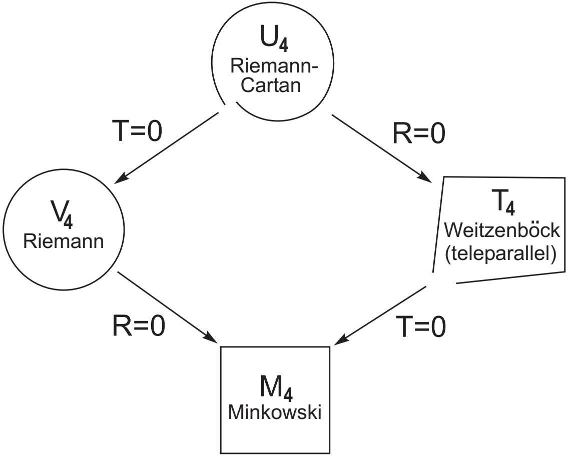
Poincaré group $P(1,3) = T(4) \rtimes S(1,3)$. The 'gravitational' potentials are

ϑ^α		orthonormal coframe (*weak* gravity = 4 potentials)
$\Gamma^{\alpha\beta}$	$= -\Gamma^{\beta\alpha}$	Lorentz connection (*strong* YM-gravity = 6 potentials)

By differentiation, we find the field strengths

$$
\begin{aligned}
T^\alpha &= D\vartheta^\alpha = d\vartheta^\alpha + \overbrace{\Gamma_\beta{}^\alpha \wedge \vartheta^\beta}^{\text{semi-dir. prod. str.}} && \text{torsion} \\
R^{\alpha\beta} &= d\Gamma^{\alpha\beta} - \underbrace{\Gamma^{\alpha\gamma} \wedge \Gamma_\gamma{}^\beta}_{\text{Lorentz gr. non-Abelian}} = -R^{\beta\alpha} && \text{curvature}
\end{aligned}
$$

The material currents of energy-momentum and spin angular momentum $(\mathfrak{T}_\alpha, \mathfrak{S}_{\alpha\beta})$ are coupled to the potentials $(\vartheta^\alpha, \Gamma^{\alpha\beta})$, respectively. As compared to GR, the additional source of gravity is the spin current $\mathfrak{S}_{\alpha\beta} = -\mathfrak{S}_{\beta\alpha}$. These 2 potentials span the geometry of spacetime: It is the Riemann-Cartan spacetime U_4. The corresponding first order Lagrangian gauge field theory is called PG. It is a framework for gravitational gauge field theories.



A Riemann-Cartan space U_4 and its different limits

Lagrangian:

$$L_{\rm total} = V(g_{\alpha\beta}\,, \vartheta^{\alpha}\,, T^{\alpha}\,, R^{\alpha\beta}) + L_{\rm matter}(g_{\alpha\beta}\,, \vartheta^{\alpha}\,, \Psi\,, \overset{\Gamma}{D}\,\Psi)\,.$$

Define the excitations (field momenta):

$$H_{\alpha} = -\frac{\partial V}{\partial T^{\alpha}}\,, \qquad H_{\alpha\beta} = -\frac{\partial V}{\partial R^{\alpha\beta}}\,,$$

Field equations:

$$\begin{aligned} DH_{\alpha} - t_{\alpha} &= \mathfrak{T}_{\alpha} && (\delta/\delta\vartheta^{\alpha}\text{: 1st field equation of gravity})\,,\\ DH_{\alpha\beta} - s_{\alpha\beta} &= \mathfrak{S}_{\alpha\beta} && (\delta/\delta\Gamma^{\alpha\beta}\text{: 2nd field equation of gravity})\,,\\ \frac{\delta L}{\delta\Psi} &= 0 && (\delta/\delta\Psi\text{: matter field equation}) \end{aligned}$$

(Einstein sector). Here energy-momentum and spin of the gauge fields are

$$\begin{aligned} t_{\alpha} &:= e_{\alpha}\rfloor V + (e_{\alpha}\rfloor T^{\beta}) \wedge H_{\beta} + (e_{\alpha}\rfloor R^{\beta\gamma}) \wedge H_{\beta\gamma}\,,\\ s_{\alpha\beta} &:= -\vartheta_{[\alpha} \wedge H_{\beta]}\,. \end{aligned}$$

Like in Maxwell and Yang-Mills, the gauge Lagrangian should be *algebraic* in T^{α} and $R^{\alpha\beta}$. Then we find 2nd order PDEs. Moreover, they should be *quadratic* in order to induce quasi-linearity of the PDEs ($\Rightarrow$ wave type eqs.):

2. Quadratic PG Lagrangian with even and odd parity terms

$$
\begin{aligned}
V_{\mathrm{PG}} \sim & \frac{1}{\kappa}\left(a_0 R + \Lambda + \sum_{I=1}^{3} a_{(I)}\,{}^{(I)}T^{\alpha} \wedge {}^{\star(I)}T_{\alpha}\right) + \frac{1}{\varrho}\sum_{I=1}^{6} r_{(I)}\,{}^{(I)}R^{\alpha\beta} \wedge {}^{\star(I)}R_{\alpha\beta} \\
& + \frac{1}{\kappa}\left(b_0 X + \sum_{1,1}^{2,3} \sigma_{(I,K)}\,{}^{(I)}T^{\alpha} \wedge {}^{(K)}T_{\alpha}\right) + \frac{1}{\varrho}\sum_{1,1;2,4}^{3,6;5,5} \mu_{(I,K)}\,{}^{(I)}R^{\alpha\beta} \wedge {}^{(K)}R_{\alpha\beta}\,.
\end{aligned}
$$

Here $X \sim \epsilon^{ijkl} R_{[ijkl]} \sim {}^{(3)}R_{\alpha\beta} \wedge \vartheta^{\alpha} \wedge \vartheta^{\beta}$. In Riemannian space, the whole 2nd line (the 'shadow' of the 1st line), with exception of $\mu(1,1)$, vanishes.

${}^{(1)}T \rightarrow$ tensor ${}^{(2)}T \rightarrow$ vector, $\sim \mathcal{V}$, ${}^{(3)}T \rightarrow$ axial vector $\sim \mathcal{A}$

${}^{(1)}R \rightarrow$ 10 Weyl, ${}^{(2)}R \rightarrow$ 9 Paircom, ${}^{(3)}R \rightarrow$ 1 Pscalar X,
${}^{(4)}R \rightarrow$ 9 Ricsymf, ${}^{(5)}R \rightarrow$ 6 Riccanti, ${}^{(6)}R \rightarrow$ 1 Scalar R.

Recently investigated by us:

$$
\begin{aligned}
V_{\mathrm{BHN}} \sim & \frac{1}{\kappa}\Big(\overbrace{a_0 R + b_0 X}^{\text{Hojman et al. 1980}} + a_2 \mathcal{V} \wedge {}^{\star}\mathcal{V} + a_3 \mathcal{A} \wedge {}^{\star}\mathcal{A} + \sigma_2 \mathcal{A} \wedge {}^{\star}\mathcal{V}\Big) \\
& + \frac{1}{\varrho}\left(r_6 R \wedge {}^{\star}R + r_3 X \wedge {}^{\star}X + \mu_3 R \wedge {}^{\star}X\right)
\end{aligned}
$$

3. Einstein-Cartan theory (EC)

Simplest Lagrangian

$$V_{\rm EC} \sim \frac{1}{\kappa} \vartheta_i{}^{\alpha} \vartheta_j{}^{\beta} \, R^{ij}{}_{\alpha\beta}(\Gamma_k{}^{\gamma\delta}) \sim \frac{1}{\kappa} \, R$$

Einstein-Cartan (EC) theory: GR plus an add. spin contact interaction,

$$\begin{aligned} \text{Ric} - \frac{1}{2}\, tr(\text{Ric}) \quad &\sim \quad \kappa \times \mathfrak{T} \sim \kappa \times \text{energy-momentum}\,, \\ \text{Tor} + 2\; tr(\text{Tor}) \quad &\sim \quad \kappa \times \mathfrak{S} \sim \kappa \times \text{spin angular momentum}\,. \end{aligned}$$

Here $\text{Ric}_{ij} := R_{kij}{}^{k}$, $R := \text{Ric}_i{}^{i}$, and κ is Einstein's gravitational constant $8\pi G/c^4$. If spin $\mathfrak{S} \;\rightarrow$ 0, then EC-theory $\rightarrow$ GR, and RC-spacetime $\rightarrow$ Riemannian spacetime. Thus, GR is included.

With $\mathfrak{S} \neq 0$, modified source of Einstein's equation: $\rho \rightarrow \rho + \kappa \mathfrak{S}^2 \;\Rightarrow$ at sufficiently high densities $\kappa \mathfrak{S}^2 \sim \rho \Rightarrow$

$$\rho_{\text{crit}} \sim m / \left(\lambda_{\text{Compton}} \ell^2_{\text{Planck}} \right) \qquad \text{(result of spin-spin contact interaction)}\,,$$

more than 10^{52} g/cm^3 or 10^{24} K for the nucleon, $\ell_{\text{crit}} \sim 10^{-26}$ cm. Spin cosmology, spin-driven inflation? For parallel Dirac spins, the contact interaction is repulsive (O'Connell). The EC-theory is a viable gravitational theory. Contact interactions in particle physics were searched for by Ellerbrock, Ph.D. thesis DESY 2004. Nothing found so far. But for EC-theory these experiments are not sensitive enough.

4. Teleparallel equivalent GR$_{||}$ of GR

Belongs to the class of translational gauge theories:

$$V_{\|} = \frac{1}{\kappa} V_{\mathrm{T}^2} + R_\alpha{}^\beta \wedge \lambda^\alpha{}_\beta \qquad (\lambda^\alpha{}_\beta = \text{ Lagrange multiplier}) ,$$

$$V_{\mathrm{T}^2} := -\frac{1}{2} T^\alpha \wedge {}^\star\Big(- \underbrace{{}^{(1)}T_\alpha}_{\text{tensor}} + 2 \underbrace{{}^{(2)}T_\alpha}_{\text{vector}} + \frac{1}{2} \underbrace{{}^{(3)}T_\alpha}_{\text{axial vector}} \Big) .$$

Viable set! Yields local Lorentz invariance $\Rightarrow$ Einstein's GR.
GR$_{||}$ in gauge $\Gamma \stackrel{*}{=} 0$, Weitzenböck spacetime, field equation is Maxwell like ($C^{ki}{}_\alpha \sim \partial^{[k}\vartheta^{i]}{}_\alpha + \cdots$):

$$D_k\, C^{ki}{}_\alpha + \text{nonlin. terms} \sim \kappa \times \mathfrak{T}_\alpha{}^i$$

$$\Box\, \vartheta^i{}_\alpha + \text{nonlin. terms} \sim \kappa \times \mathfrak{T}_\alpha{}^i \quad \text{(in Hilbert gauge)}$$

Compare Einstein's equation ($g_{ij} = g_{ji}$):

$$\Box\, g_{ij} + \text{nonlin. terms} \sim \kappa \times \mathfrak{t}_{ij} \quad \text{(in Hilbert gauge)}$$

For scalar and for Maxwell matter, that is, for $\mathfrak{T}_{ij} = \mathfrak{t}_{ij}$, it can be shown that GR$_{||}$ and GR are equivalent. This suggests already that the answer to the title question should be affirmative.

5. Abelian and non-Abelian gauge field ths. comp.

With the excitatation $H = H(\mathcal{D}, \mathcal{H})$ and the field strength $F = F(E, B)$:

Maxwell:

$$\boxed{dH = \mathfrak{J}}\,, \qquad dF = 0\,, \qquad H = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,{}^\star F\,, \qquad d\mathfrak{J} = 0\,.$$

Yang-Mills:

$$\boxed{\overset{A}{D}\, H = \mathfrak{J}}\,, \qquad \overset{A}{D}\, F = 0\,, \qquad H = \alpha_0\,{}^\star F\,, \qquad \overset{A}{D}\, \mathfrak{J} = 0\,.$$

$$dH + A \wedge H = \mathfrak{J}\,, \quad dF + A \wedge F = 0\,, \quad d\mathfrak{J} - A \wedge \mathfrak{J} = 0\,,$$

$$\overset{A}{\mathfrak{J}} := -A \wedge H, \quad \text{with} \quad \check{\mathfrak{J}} := \mathfrak{J} + \overset{A}{\mathfrak{J}} \quad \text{and} \quad d\check{\mathfrak{J}} \cong 0\,.$$

Poincaré:

$$\boxed{\overset{\Gamma}{D}\, H_\alpha - t_\alpha = \mathfrak{T}_\alpha}\,, \qquad \overset{\Gamma}{D}\, T^\alpha = R_\beta{}^\alpha \wedge \vartheta^\beta\,,$$

$$H_\alpha = H_\alpha(T^\gamma, R^{\gamma\delta})\,, \quad \overset{\Gamma}{D}\, \mathfrak{T}_\alpha = (e_\alpha \rfloor T^\beta) \wedge \mathfrak{T}_\beta + (e_\alpha \rfloor R^{\beta\gamma}) \wedge \mathfrak{S}_{\beta\gamma};$$

$$\boxed{\overset{\Gamma}{D}\, H_{\alpha\beta} - s_{\alpha\beta} = \mathfrak{S}_{\alpha\beta}}\,, \qquad \overset{\Gamma}{D}\, R^{\alpha\beta} = 0\,,$$

$$H_{\alpha\beta} = H_{\alpha\beta}(T^\gamma, R^{\gamma\delta})\,, \qquad \overset{\Gamma}{D}\, \mathfrak{S}_{\alpha\beta} - \vartheta_{[\alpha} \wedge \mathfrak{T}_{\beta]} = 0\,.$$

Translational and rotational gauge currents induced by the universality of gravity. The gauge potentials ϑ^α and $\Gamma^{\alpha\beta}$ carry tensorial charges $t_\alpha, s_{\alpha\beta}$.

The Yang-Mills potential A carries also an own isospin current, namely $\overset{A}{\mathfrak{J}}$, but it is *not* tensorial.

Schematically, it looks as follows:

Maxwell:

$$\mathbf{d}\,[U(1)\text{ field strength}] \sim U(1)\text{-current}\,.$$

Yang–Mills:

$$\overset{A}{\mathbf{D}}\,[SU(2)\text{ field strength}] \sim SU(2)\text{-current}\,.$$

Poincaré:

$$\begin{aligned}\overset{\Gamma}{\mathbf{D}}\,[\text{transl. field strength}] - \text{transl. gauge current} \quad &\sim \quad \text{transl. current}\,,\\ \overset{\Gamma}{\mathbf{D}}\,[\text{rotat. field strength}] - \text{rotat. gauge current} \quad &\sim \quad \text{rotat. current}\,.\end{aligned}$$

Einstein-Cartan (as degenerate PG):

$$\begin{aligned}\text{rotat. field strength} \quad &\sim \quad \text{transl. current}\,,\\ \text{transl. field strength} \quad &\sim \quad \text{rotat. current}\,.\end{aligned}$$

$$\begin{aligned}\text{lhs-first}(\partial\Gamma,\Gamma,\vartheta) \quad &\sim \quad \mathfrak{T}\,,\\ \text{lhs-second}(\partial\vartheta,\vartheta,\Gamma) \quad &\sim \quad \mathfrak{S}\,.\end{aligned}$$

6. EC-theory and the Freud superpotential $\mathcal{F}_\alpha$

EC is in many respects degenerate, see last slide. A contact interaction must become of finite, if very small range $\Rightarrow$ massive Lorentz gauge bosons, compare Fermi's weak interaction theory and the W and Z.
Field equations of EC are algebraic in $R^{\alpha\beta}$ and T^α, respectively. In spite of this, we want to try to put them in a form reminiscent of Yang-Mills type field equations ($\kappa = 1$):

$$\underbrace{G_\alpha}_{\text{Einstein 3-form}} := \tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} = \mathfrak{T}_\alpha \qquad \Longrightarrow \qquad d\mathcal{F}_\alpha - t'_\alpha = \mathfrak{T}_\alpha \,,$$

$$\underbrace{P_{\alpha\beta}}_{\text{Palatini 3-form}} := \tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge T^{\gamma} = \mathfrak{S}_{\alpha\beta} \qquad \Longrightarrow \qquad d(\tfrac{1}{2}\eta_{\alpha\beta}) - s'_{\alpha\beta} = \mathfrak{S}_{\alpha\beta} \,.$$

The η-basis is defined in the conventional way: If we take the interior product $\rfloor$ of an arbitrary frame e_α with the *metric volume element* 4-form η, then we find a 3-form η_α; if we contract again, we find a 2-form $\eta_{\alpha\beta}$, etc.:

$$\begin{aligned}
\eta_\alpha &:= e_\alpha \rfloor \eta = \tfrac{1}{6}\, \eta_{\alpha\beta\gamma\delta}\, \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta \,, \\
\eta_{\alpha\beta} &:= e_\beta \rfloor \eta_\alpha = \tfrac{1}{2}\, \eta_{\alpha\beta\gamma\delta}\, \vartheta^\gamma \wedge \vartheta^\delta \,, \\
\eta_{\alpha\beta\gamma} &:= e_\gamma \rfloor \eta_{\alpha\beta} = \eta_{\alpha\beta\gamma\delta}\, \vartheta^\delta \,, \\
\eta_{\alpha\beta\gamma\delta} &= e_\delta \rfloor \eta_{\alpha\beta\gamma} = e_\delta \rfloor e_\gamma \rfloor e_\beta \rfloor e_\alpha \rfloor \eta \,.
\end{aligned}$$

The coframe ϑ^β is dual to the frame e_α, that is, $e_\alpha \rfloor \vartheta^\beta = \delta_\alpha^\beta$. We need also the exterior covariant derivatives of the eta-forms:

$$\begin{aligned}
D\eta_\alpha &= T^\delta \wedge \eta_{\alpha\delta}\,, \\
D\eta_{\alpha\beta} &= T^\delta \wedge \eta_{\alpha\beta\delta}\,, \\
D\eta_{\alpha\beta\gamma} &= T^\delta \wedge \eta_{\alpha\beta\gamma\delta}\,, \\
D\eta_{\alpha\beta\gamma\delta} &= 0\,.
\end{aligned}$$

If one desires to introduce a superpotential à la Freud (1939), then one has to substitute $R^{\beta\gamma}$ into the 1st field eq. and one of the above formlas into the 2nd field eq.:

$$\begin{aligned}
\tfrac{1}{2}\,\eta_{\alpha\beta\gamma} \wedge \left(d\Gamma^{\beta\gamma} - \Gamma^{\beta\delta} \wedge \Gamma_\delta{}^\gamma\right) &= \mathfrak{T}_\alpha\,, \\
\tfrac{1}{2}\,D\eta_{\alpha\beta} = d(\tfrac{1}{2}\eta_{\alpha\beta}) + \Gamma_{[\alpha}{}^\gamma \wedge \eta_{\beta]\gamma} &= \mathfrak{S}_{\alpha\beta}\,.
\end{aligned}$$

We partially integrate the first term and find immediately,

$$d\,(\underbrace{-\frac{1}{2}\,\eta_{\alpha\beta\gamma} \wedge \Gamma^{\beta\gamma}}_{\mathcal{F}_\alpha :=}) + \frac{1}{2}\,(d\eta_{\alpha\beta\gamma}) \wedge \Gamma^{\beta\gamma} - \frac{1}{2}\,\eta_{\alpha\beta\gamma} \wedge \Gamma^{\beta\delta} \wedge \Gamma_\delta{}^\gamma = \mathfrak{T}_\alpha\,.$$

We define the Freud superpotential 2-form $\mathcal{F}^\alpha$ and the 3-form t'_α as

$$\begin{aligned} \mathcal{F}_\alpha &:= -\tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge \Gamma^{\beta\gamma}\,, \\ t'_\alpha &:= -\tfrac{1}{2}\left(d\eta_{\alpha\beta\gamma}\right) \wedge \Gamma^{\beta\gamma} + \tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge \Gamma^{\beta\delta} \wedge \Gamma_\delta{}^\gamma\,. \end{aligned}$$

We recall $D\eta_{\alpha\beta\gamma} = T^\mu \wedge \eta_{\alpha\beta\gamma\mu}$. As a consequence

$$d\eta_{\alpha\beta\gamma} = 3\,\Gamma_{[\alpha}{}^\delta \wedge \eta_{\beta\gamma]\delta} + T^\delta \wedge \eta_{\alpha\beta\gamma\delta}\,.$$

After some algebra we find eventually

$$t'_\alpha = \eta_{\beta\gamma[\alpha} \wedge \Gamma_{\delta]}{}^\beta \wedge \Gamma^{\delta\gamma} - \tfrac{1}{2}\eta_{\alpha\beta\gamma\delta}\,\Gamma^{\beta\gamma} \wedge T^\delta\,.$$

Then we can rewrite the 1st and the 2nd EC field equations simply as

$$\boxed{d\,\mathcal{F}_\alpha - t'_\alpha = \mathfrak{T}_\alpha}\,, \qquad \boxed{d(\tfrac{1}{2}\eta_{\alpha\beta}) + \Gamma_{[\alpha}{}^\gamma \wedge \eta_{\beta]\gamma} = \mathfrak{S}_{\alpha\beta}}\,.$$

This is the quasi-Yang-Mills or quasi-PG form of the field equations of the EC-theory (Why only quasi? Do you know?...) As before, we define the energy-momentum and spin complexes

$$\check{\mathfrak{T}}_\alpha := t'_\alpha + \mathfrak{T}_\alpha\,, \qquad \check{\mathfrak{S}}_{\alpha\beta} := -\Gamma_{[\alpha}{}^\gamma \wedge \eta_{\beta]\gamma} + \mathfrak{S}_{\alpha\beta}\,.$$

Consquently, we find energy-momentum and angular momentum laws

$$d\,\mathcal{F}_\alpha = \check{\mathfrak{T}}_\alpha\,, \qquad d(\tfrac{1}{2}\eta_{\alpha\beta}) = \check{\mathfrak{S}}_\alpha \qquad \text{with} \qquad d\,\check{\mathfrak{T}}_\alpha = 0\,, \qquad d\,\check{\mathfrak{S}}_{\alpha\beta} = 0\,.$$

The plan of my seminar is to concentrate on GR. We could go on also in the EC context, but most of you are probably mainly interested in GR. For now on we put the matter spin to zero: $\mathfrak{S}_{\alpha\beta} = 0$!. Thus, torsion $T^\alpha = 0$ in my future considerations.
The appearance of $\mathcal{F}_\alpha$ is the same as before, but the connection is now a Riemann/Levi-Civita connection $\widetilde{\Gamma}^{\alpha\beta}$. Hence $d\,\mathcal{F}_\alpha - t'_\alpha = \mathfrak{T}_\alpha$ with

$$\mathcal{F}_\alpha = -\tfrac{1}{2}\eta_{\alpha\beta\gamma} \wedge \widetilde{\Gamma}^{\beta\gamma} = \tfrac{1}{2}\mathcal{F}_{ik\alpha}\, dx^i \wedge dx^k\,, \qquad t'_\alpha = \eta_{\beta\gamma[\alpha} \wedge \widetilde{\Gamma}_{\delta]}{}^\beta \wedge \widetilde{\Gamma}^{\delta\gamma}\,.$$

Freud found his superpotential[1] in 1939 as the affine tensor density

$$\mathfrak{A}^{in}{}_k = \tfrac{1}{2}\begin{vmatrix} \delta^i_k & \delta^n_k & \delta^\mu_k \\ \mathfrak{g}^{i\rho} & \mathfrak{g}^{n\rho} & \mathfrak{g}^{\mu\rho} \\ \Gamma^i_{\rho\mu} & \Gamma^n_{\rho\mu} & \Gamma^\mu_{\rho\mu} \end{vmatrix} = -\mathfrak{A}^{ni}{}_k\,.$$

$\mathcal{F}_{ik\alpha} = -\mathcal{F}_{ki\alpha}$ and $\mathfrak{A}^{in}{}_k$ are, apart from conventions, the same quantities.

[1] Ph. Freud, *On the expressions of total energy and total momentum of a material system in general relativity theory* (in German), Annals of Mathematics **40**, 417–419 (1939).

7. Transform the Freud superpotential $\mathcal{F}_\alpha$ into ${}^\star C^\alpha$

We want to show that the Einstein equation is a translational gauge field equation. We eliminate now the connetion in terms of derivatives of the coframe and the metric. We recall the object of anholonom(it)y $C^\alpha := d\vartheta^\alpha$:

$$\widetilde{\Gamma}_{\alpha\beta} := \frac{1}{2}dg_{\alpha\beta} + (e_{[\alpha}\rfloor dg_{\beta]\gamma})\vartheta^\gamma + e_{[\alpha}\rfloor C_{\beta]} - \frac{1}{2}(e_\alpha\rfloor e_\beta\rfloor C_\gamma)\vartheta^\gamma\,.$$

We used *orthonormal* frames; then, the first two terms on the rhs vanish. Accordingly, the Freud's superpotential becomes

$$\mathcal{F}_\alpha = -\tfrac{1}{2}\eta_\alpha{}^{\beta\gamma}\wedge\left[e_\beta\rfloor C_\gamma - \tfrac{1}{2}(e_\beta\rfloor e_\gamma\rfloor C_\delta)\vartheta^\delta\right]$$

Taking this is consideration, the Einstein equation $d\,\mathcal{F}_\alpha - t'_\alpha = \mathfrak{T}_\alpha$ is a 2nd order PDE in the $\vartheta^\alpha = \vartheta_i{}^\alpha dx^i$. Since ϑ^α is the translation potential, we found a Yang-Mills type equation for the translational potential. And this is what Einstein's equation is.

A Yang-Mills type equation is $dH - t' = \mathfrak{T}$, with $H \sim {}^\star F$. Our translation field strength here is $C^\alpha = d\vartheta^\alpha = \frac{1}{2} C_{ik}{}^\alpha \, dx^i \wedge dx^k$. This 24 component quantity can be decomposed into 3 irreducible pieces:

$$C^\alpha = {}^{(1)}C^\alpha + {}^{(2)}C^\alpha + {}^{(3)}C^\alpha$$

with $16 + 4 + 4$ components, respectively. In $SU(2)$ Yang-Mills the field strength is irreducible!
Purely gemetrical manipulation yield, after some heavy algebra, the formula (the overall sign needs to be rechecked!):

$$\mathcal{F}_\alpha := {}^\star(-{}^{(1)}C_\alpha + 2\,{}^{(2)}C_\alpha + \tfrac{1}{2}\,{}^{(3)}C_\alpha).$$

Our end result is then the Einstein equation written in coframes:

$$\boxed{d\,{}^\star\left(-{}^{(1)}C_\alpha + 2\,{}^{(2)}C_\alpha + \tfrac{1}{2}\,{}^{(3)}C_\alpha\right) - t'_\alpha = \mathfrak{T}_\alpha\,.}$$

All the 2nd derivatives of the coframe are contained within the parentheses $(\quad)$. It can be shown that t'_α corresponds to the tranlational part of the PG energy gauge current t_α defined on slide # 4. All the derivations leading to our end result have been exact; there was no approximation involved.

8. Discussion

- Yes, GR is a translational gauge theory in disguise.
- Questions?
- Thank you for your attention and your patience!
- Milutin, all the best to you and your family and remain healthy and active in 3d and 4d!

Soli Deo Gloria