Introduction to Noncommutative Geometry

1. General introduction and motivation

2. Models with finite-dimensional algebras, with and without commutative limits

3. Introduction to differential calculi

4. Introduction to connections

5. Introduction to metrics and compatible linear connections

6. Models with infinite-dimensional algebras

1. General introduction and motivation

History:

Heisenberg *et al.* and the lattice (1930's)

Snyder and 'fuzz' (1947); Lorentz invariance

von Neumann and 'noncommutative geometry' (1950's)

Connes and 'noncommutative differential geometry' (1980's)

The simplest expression of Snyder's idea:

Consider S^2 with an action of SO_3; choose a lattice: the north and south poles; this breakes SO_3 invariance

The lattice algebra of functions $= \mathbb{C} \times \mathbb{C}$; extend to $M_2(\mathbb{C})$ to recover the invariance; the two points become two 'cells'; an 'observable' is an hermitian 2×2 matrix, with two real eigenvalues, its values on the two 'cells'

Similar to replacing a classical spin which can take two values by a quantum spin of total spin $1/2$; only the latter is invariant under SO_3

General s with $n = 2s + 1$; replace M_2 by M_n; there are n cells of area $2\pi\kbar$ with

$$n \simeq \frac{\mathrm{Vol}(S^2)}{2\pi\kbar}, \quad [\kbar] = (\mathrm{length})^2$$

Mathematics:

Formulate as much as possible the geometry of a manifold V in terms of an algebra $\mathcal{C}(V)$ of complex-valued functions (smooth, continuous, measurable, all functions) (Koszul 1960's)

Replace the algebra $\mathcal{C}(V) \mapsto \mathcal{A}$ by a noncommutative algebra $\mathcal{A}$ (associative, with unit and with an involution $*$)

Since V as a manifold of dimension m can be embedded in $\mathbb{R}^n$ for some $n > m$, choose $\mathcal{A}$ defined in terms of n generators and $n - m$ relations

Represent the algebra $\mathcal{A}$ by an algebra of operators on a Hilbert space

Physics:

Practical physics: introduce a cut-off Λ; points 'fuzzy' to order Λ^{-1}

'Fundamental' Physics: replace points by 'Planck cells'; no UV divergences

Solid-state analogy: Coordinates as order parameters; course-graining

Related things: random lattices, qantum nets, twistors, Sakarov's induced gravity, Wheeler's graviton as phonon *et cætera*

Unrelated things: Schrödinger's position operators

$$\mathbf{q}(t) = \mathbf{x}(t) + m^{-2}\mathbf{S} \times \mathbf{p}(t)$$

to explain the Zitterbewegung of an electron

Comparison with quantum mechanics: particle in a plane: (x^1, x^2, p_1, p_2)

a) Classical mechanics: 4 commuting operators

b) Quantum mechanics: $[x^i, p_j] = \hbar\delta^i_j$;
'Bohr cells' of area $2\pi\hbar$

c) Magnetic field B normal to the plane:
$[p_1, p_2] = i\hbar eB$; 'Landau cells' of area $\hbar eB$;
IR cut-off: $p^2 \gtrsim \hbar eB$

d) Gaussian curvature K: $p^2 \gtrsim \hbar^2 K$

e) 'Quantized' coordinates: $x^i \mapsto q^i$;
$[q^1, q^2] = i\kbar q^{12}$; 'Planck cells' of area $2\pi\kbar$;
UV cut-off: $p^2 \lesssim \hbar^2/\kbar$

Situations d) and e) together imply:

$$I = \int \frac{dp_1 dp_2}{p_1^2 + p_2^2} \sim \log(\kbar K)$$

Let ϕ_r be eigenmodes of an operator Δ,
$\Delta\phi_r = \lambda_r\phi_r$, which span a Hilbert space $\mathcal{H} \subset \mathcal{A}$:

$$\mathrm{Tr}\,(\phi_r^*\phi_s) = \delta_{rs}, \qquad \phi = \sum_r \phi_r \mathrm{Tr}\,(\phi_r^*\phi)$$

The 2-point function $G \in \mathcal{H} \otimes \mathcal{H}$:

$$G = \sum_r \lambda_r^{-1}\phi_r \otimes \phi_r^*$$

Suppose $\mathcal{A} = \mathcal{A}(q^1, q^2)$ and rewrite

$$q^\mu \otimes 1 = \bar{q}^\mu + \delta q^\mu, \qquad 1 \otimes q^\mu = \bar{q}^\mu - \delta q^\mu$$

Then if q^{12} is in the center of the algebra

$$[\bar{q}^\mu, \delta q^\nu] = 0, \quad [\bar{q}^1, \bar{q}^2] = [\delta q^1, \delta q^2] = \frac{1}{2} i\hbar q^{12}$$

There can be no state $|0\rangle$ (on the diagonal) with

$$\delta q^1|0\rangle = 0, \quad \delta q^2|0\rangle = 0$$

Complication: q^{12} need not lie in the center

Conjecture to determine q^{12}:

it determines (in Wheeler's language) the 'lattice' spacings away from (flat space) equilibrium;
it is in 1-1 correspondence with the classical 'gravitational' field

As examples we shall find that (after addition of a differential calculus) if

a) $q^{12} = q^3$, $\sum (q^i)^2 = r^2$: the surface is a sphere

b) $q^{12} = 1$: the surface is flat

c) $\bar{k} q^{12} = -2hq^2$: the surface is a pseudosphere

In phase space the Jacobi identities imply:

$$[q^i, p_j] = i\hbar\delta^i_j + i\bar{k}A^i_j$$

A noncommutative vision of gravity:

The euclidean classical action:

$$S[g] = \Lambda_c \mathrm{Vol}(V)[g] + \mu_P^2 \int_V R + \cdots$$

The euclidean quantum action:

$$\Gamma[g] = S[g] + \frac{1}{2}\hbar \mathrm{Tr} \log \Delta[g] \simeq z_0 \mu_P^4 \mathrm{Vol}(V)[g]) +$$

$$z_1 \mu_P^2 \int_V R + z_2 (\log \frac{\mu_P}{\mu}) S_2[g] + 0(\hbar^2) + \cdots$$

$\Delta[g]$: any mode in a gravitational field g

Sakharov's idea: there is no classical action

Wheeler's idea: 'Gravitation is to particle physics as elasticity is to chemical physics: merely a statistical measure of residual energies.'

Noncommutative version: Gravitation is a measure of a variation of the spectral distribution of some operator away from an 'equilibrium' value

A typical model:

Replace (Minkowski) coordinates x^μ by generators q^μ of a noncommutative algebra $\mathcal{A}_{\kbar}$ with

$$[q^\mu, q^\nu] = i\kbar q^{\mu\nu}, \qquad \kbar \simeq \mu_P^{-2} = G\hbar$$

Structure of the algebra: $[q^\lambda, q^{\mu\nu}]$ *et cætera*

'Heisenberg' uncertainty relations: $\Lambda^2 \kbar \lesssim 1$

Fuzzy space-time: cells of volume $\simeq (2\pi\kbar)^2$

In the limit $\mu_P \to \infty$: $q^\mu \to x^\mu$ or perhaps:
Singular 'renormalization constant': $q^\mu \to z\, x^\mu$

Representation: q^μ become unbounded hermitian operators on some Hilbert space

The whole idea is contained in the diagram

$$\begin{array}{ccc} \mathcal{A}_{\kbar} & \Longleftarrow & \Omega^*(\mathcal{A}_{\kbar}) \\ \Downarrow & & \Uparrow \\ \text{Cut-off} & & \text{Gravity} \end{array}$$

2. Finite-dimensional models

Consider the decomposition $\mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C}^{n-m}$
Then

$$M_n = \begin{pmatrix} M_m & M_n^{-\prime} \\ M_n^{-\prime\prime} & M_{n-m} \end{pmatrix} = M_n^+ \oplus M_n^-$$

with $M_n^+ = M_m \times M_{n-m}, \quad M_n^- = M_n^{-\prime} \cup M_n^{-\prime\prime}$

Examples: $n = 2,\ m = 1, \quad n = 3,\ m = 1$

$$M_n = M_n(x^a) =$$
$$\{(x^a) \mid x^a = ƛr^{-1}J^a,\ [J_a, J_b] = i\epsilon_{abc}J^c,$$
$$J_a J^a = (n^2 - 1)/4 = r^4/ƛ^2\}$$

From Casimir relation: $g_{ab}x^a x^b = r^2, \quad n \simeq \dfrac{4\pi r^2}{2\piƛ}$

$$M_n = \mathcal{A}_{l/n} = M_n(u, v) =$$
$$\{(u, v) \mid u^n = v^n = 1,\ uv = qvu,$$
$$q = e^{2\pi i\alpha},\ \alpha = l/n\}$$

Representation of $\mathcal{A}_{1/n}$: $\{|j\rangle_1\},\ \{|k\rangle_2\} \in \mathbb{C}^n$

$$u|j\rangle_1 = q^j|j\rangle_1, \qquad v|j\rangle_1 = |j+1\rangle_1,$$
$$u|k\rangle_2 = |k-1\rangle_2, \quad v|k\rangle_2 = q^k|k\rangle_2$$

'Fourier' transformation:

$$|j\rangle_1 = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} q^{+jk}|k\rangle_2, \quad |k\rangle_2 = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} q^{-jk}|j\rangle_1$$

One deduces immediately the relations

$$uv = qvu, \qquad u^n = 1, \qquad v^n = 1, \qquad q^n = 1$$

Define hermitian matrices x and y by

$$x|j\rangle_1 = \frac{\hbar}{r} j|j\rangle_1, \qquad y|k\rangle_2 = \frac{\hbar}{r} k|k\rangle_2$$

One finds that

$$u = e^{ix/r},\ v = e^{iy/r}, \quad q = e^{i\hbar/r^2}, \quad n = \frac{(2\pi r)^2}{2\pi\hbar}$$

The algebra M_n has derivations

$\mathrm{Der}(M_n) =$

$$\{X : M_n \to M_n \mid X(fg) = Xfg + fXg\}$$

For example: $\quad e_1 = \frac{1}{i\hbar}\mathrm{ad}\, y, \quad e_2 = -\frac{1}{i\hbar}\mathrm{ad}\, x$

$$\begin{aligned} &e_1 u = ir^{-1}u(1 - nP_2), \quad && e_1 v = 0, \\ &e_2 u = 0, && e_2 v = ir^{-1}v(1 - nP_1) \end{aligned}$$

with $P_1 = |0\rangle_2\langle 0|, \quad P_2 = |n-1\rangle_1\langle n-1|$

One finds $\quad e_1 u^n = 0, \quad e_2 v^n = 0, \quad [e_1, e_2] = 0$

Recall 2-torus with $\tilde{u} = e^{i\tilde{x}/r},\ \tilde{v} = e^{i\tilde{y}/r}$ and $\tilde{e}_1 f = \partial_{\tilde{x}} f,\ e_2 f = \partial_{\tilde{y}} f$:

$$\begin{aligned} &\tilde{e}_1 \tilde{u} = ir^{-1}\tilde{u}, \quad && \tilde{e}_1 \tilde{v} = 0, \\ &\tilde{e}_2 \tilde{u} = 0, && \tilde{e}_2 \tilde{v} = ir^{-1}\tilde{v} \end{aligned}$$

With lattice: $\tilde{u}^n = 1,\ \tilde{v}^n = 1$ but no derivations

3. Differential calculi

Consider associative $\mathcal{A}$ and a graded algebra

$$\Omega^*(\mathcal{A}) = \bigoplus_{i \geq 0} \Omega^i(\mathcal{A}), \quad \Omega^0(\mathcal{A}) = \mathcal{A}$$

direct sum of a family of $\mathcal{A}$-bimodules; if the grading is a $\mathbb{Z}_2$-grading we write $\Omega^+(\mathcal{A}) = \mathcal{A}$

A differential d is a graded derivation of $\Omega^*(\mathcal{A})$ with $d^2 = 0$; if $\alpha \in \Omega^i(\mathcal{A})$ and $\beta \in \Omega^j(\mathcal{A})$ then $\alpha\beta \in \Omega^{i+j}(\mathcal{A})$ and $d(\alpha\beta) \in \Omega^{i+j+1}(\mathcal{A})$ with

$$d(\alpha\beta) = d\alpha\beta + (-1)^i \alpha d\beta$$

Differential algebra = a graded algebra with d

We say $\Omega^*(\mathcal{A})$ is a differential calculus over $\mathcal{A}$

Universal calculus: $\Omega^*_u(\mathcal{A})$

There exists a construction uniquely defined by the bimodule $\Omega^1(\mathcal{A})$

Define the map $\mathcal{A} \xrightarrow{d_u} \mathcal{A} \otimes \mathcal{A}$ by

$$d_u f = 1 \otimes f - f \otimes 1$$

Define $\Omega^1_u(\mathcal{A}) \subset \mathcal{A} \otimes \mathcal{A}$ image of d_u; for $\Omega^1(\mathcal{A})$ another bimodule of 1-forms define

$$\Omega^1_u(\mathcal{A}) \xrightarrow{\phi_1} \Omega^1(\mathcal{A})$$

by

$$\phi_1(d_u f) = df$$

Because $d1 = 0$ the map is well defined; we have

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{d_u} & \Omega^1_u(\mathcal{A}) \\ \| & & \phi_1 \downarrow \\ \mathcal{A} & \xrightarrow{d} & \Omega^1(\mathcal{A}) \end{array}$$

We can write $\Omega^1(\mathcal{A}) = \Omega^1_u(\mathcal{A})/\mathrm{Ker}\,\phi_1$

Every bimodule of 1-forms can be so written

Product: $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\pi} \Omega^2(\mathcal{A})$

Example: let $\mathcal{A} = \mathcal{C}(V)$ and $\Omega^1(\mathcal{A}) \equiv \Omega^1(V)$

If $f \in \mathcal{A}$ then $d_u f$ is the function of 2 variables

$$d_u f(x, y) = f(y) - f(x)$$

The de Rham 1-form: $df = \partial_\lambda f dx^\lambda$

Expand the function $f(y)$ about the point x:

$$f(y) = f(x) + (y^\lambda - x^\lambda)\partial_\lambda f + \cdots$$

The map ϕ_1 is given by

$$\phi_1(y^\lambda - x^\lambda) = dx^\lambda$$

It annihilates $f(x, y) \in \Omega^1_u(\mathcal{A})$ 2nd order in $x - y$

One such form is $f d_u g - d_u g f$:

$$(f d_u g - d_u g f)(x, y) = -(f(y) - f(x))(g(y) - g(x))$$

It does not vanish in $\Omega^1_u(\mathcal{A})$ but its image in
$\Omega^1(\mathcal{A})$ under ϕ_1 is equal to zero

The Dirac operator $\not{D}\psi = i\gamma^\alpha D_\alpha \psi, \;\; \psi \in \mathcal{H}$

$$\not{D} = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

$$\not{D}\psi = D^+\psi^+ + D^-\psi^-, \quad D^\pm\psi^\pm \in \mathcal{H}^\mp$$

Moving frame e_α with $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$:

$$\not{D}(f\psi) = (ie_\alpha f)\gamma^\alpha\psi + f\not{D}\psi$$

and therefore

$$e_\alpha f \gamma^\alpha = -i[\not{D}, f]$$

Map $\gamma^\alpha \mapsto \theta^\alpha$; write

$$\hat{d}f = e_\alpha f \theta^\alpha = -i[\not{D}, f]$$

If the commutator is taken to be graded we have

$$\hat{d}^2 f = -[\not{D}^2, f], \qquad \hat{d}^2 \neq 0$$

The set $(\mathcal{A}, \mathcal{H}, \not{D})$ is called a spectral triple

Example: write $\mathbb{C}^2 = \mathbb{C}^1 \oplus \mathbb{C}^1$ and decompose

$$M_2 = M_2^+ \oplus M_2^-$$

The commutative algebra M_2^+ is the algebra of functions on 2 points

Graded derivation $\hat{d}\alpha$ of $\alpha \in M_n$:

$$\hat{d}\alpha = -[\eta, \alpha], \qquad \eta \in M_n^-$$

The bracket is graded and η is antihermitian

We find $\hat{d}\eta = -2\eta^2$ and $\hat{d}^2\alpha = [\eta^2, \alpha]$

Set $\eta^2 = -1$: $\hat{d} \equiv d$, $d^2 = 0$;
$\Omega_\eta^* = M_2$ is a differential calculus over M_2^+

Since for all p: $\Omega_\eta^{2p} = M_2^+$, $\Omega_\eta^{2p+1} = M_2^-$
we can identify

$$\Omega_\eta^* = \Omega_\eta^+ \oplus \Omega_\eta^-, \qquad \Omega_\eta^\pm = M_n^\pm$$

Notice that $d\eta + \eta^2 = 1$

Spectral triple: $(M_2^+, \mathbb{C}^2, i\eta)$

Example: write $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}^1$ and decompose

$$M_3 = M_3^+ \oplus M_3^-$$

The algebra $M_3^+ = M_2 \times M_1 \quad \sim \quad$ functions on 2 points with an extra structure on one

Graded derivation $\hat{d}\alpha = -[\eta, \alpha]$ of $\alpha \in M_n$ with

$$\eta = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ -a_1^* & -a_2^* & 0 \end{pmatrix} \in M_n^-$$

We have $\Omega_\eta^0 = M_3^+, \quad \Omega_\eta^1 = M_3^-$

It is not possible to have $\hat{d}^2 = 0$; define

$$\Omega_\eta^2 = M_3^+ / \mathrm{Im}\, \hat{d}^2 = M_1, \quad \Omega_\eta^p = 0, \ p \geq 3$$

Ω_η^* is a differential calculus over M_3^+

Notice that again $\quad d\eta + \eta^2 = 1$

Spectral triple: $\quad (M_3^+, \mathbb{C}^3, i\eta)$

Example: M_n with basis λ_a:

$$\lambda_a \lambda_b = \frac{1}{2} C^c{}_{ab} \lambda_c + \frac{1}{2} D^c{}_{ab} \lambda_c - \frac{1}{n} g_{ab}$$

Killing metric: g_{ab}; structure constants: $C^c{}_{ab}$

Contruct $\Omega^*(M_n)$ with generators θ^a and relations

$$f\theta^b = \theta^b f, \qquad \theta^a \theta^b = -\theta^b \theta^a$$

and a differential defined by

$$d\lambda^a = C^a{}_{bc}\, \lambda^b \theta^c, \qquad d\theta^a = -\frac{1}{2} C^a{}_{bc}\, \theta^b \theta^c$$

Special 1-form: $\theta = -\lambda_a \theta^a = -\frac{1}{n} \lambda_a d\lambda^a$

From the definitions $df = -[\theta, f]$

Notice that $d\theta + \theta^2 = 0$

Spectral triple: $(M_n, \mathbb{C}^n, i\theta)$

4. Yang-Mills connections

Connection $\equiv$ covariant derivative

Left connection (Yang-Mills) on left $\mathcal{A}$-module $\mathcal{H}$:

$$\mathcal{H} \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}$$

with a left Leibniz rule

$$D(f\psi) = df \otimes \psi + f D\psi, \qquad f \in \mathcal{A}, \quad \psi \in \mathcal{H}$$

Extension: $\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \xrightarrow{D} \Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}$ by

$D(\alpha \otimes \psi) = d\alpha \otimes \psi + (-1)^n \alpha \otimes D\psi, \quad \alpha \in \Omega^n(\mathcal{A})$

We shall drop the '$\otimes$' symbol

In particular one verifies that

$$D^2(f\psi) = f D^2\psi$$

Define $\mathrm{Curv}(\psi) = D^2\psi$

Example with differential calculus Ω^*_η over M_3^+ and with $\mathcal{H}$ the bimodule M_3^+:

Covariant derivative: $D_{(0)}\psi = -\eta\psi$

In fact: $D_{(0)}(f\psi) = -\eta f\psi = -f\eta\psi + df\psi$

General case: $D\psi = -\eta\psi - \psi\phi$

One can write $D\psi = d\psi + \omega\psi$ in terms of a 'connection form' ω which transforms as

$$\omega' = g^{-1}\omega g + g^{-1}dg, \quad g \in U_2 \times U_1$$

In particular: $\eta' = \eta$; therefore

$$\omega = \eta + \phi, \qquad \phi' = g^{-1}\phi g$$

Curvature: $\Omega = d\omega + \omega^2 = 1 + \phi^2 = 1 - |\phi|^2$

Action: $V(\phi) = \frac{1}{4}\mathrm{Tr}\,(1 - |\phi|^2)^2$

The electromagnetic action on the 'space' M_3^+

Example with differential calculus $\Omega^*(M_n)$ over M_n and with $\mathcal{H}$ the bimodule M_n:

Covariant derivative: $D_{(0)}\psi = -\theta\psi$

General case: $D\psi = -\theta\psi - \psi\phi$

In terms of a 'connection form'

$$\omega' = g^{-1}\omega g + g^{-1}dg, \quad g \in U_n$$

In particular: $\theta' = \theta$; therefore

$$\omega = \theta + \phi, \qquad \phi' = g^{-1}\phi g$$

Curvature: $\Omega = d\omega + \omega^2 = \frac{1}{2}\Omega_{ab}\theta^a\theta^b$

where $\Omega_{ab} = [\phi_a, \phi_b] - C^c{}_{ab}\,\phi_c$

$C^c{}_{ab}$ is a 'Christoffel symbol'

Action: $V(\phi) = \frac{1}{4}\mathrm{Tr}\,(\Omega_{ab}\Omega^{ab})$

The electromagnetic action on the 'space' M_n

5. Metrics and linear connections

Let $\mathcal{M}$ an $\mathcal{A}$-bimodule and $\Omega^*(\mathcal{A})$ a differential calculus; covariant derivative

$$\mathcal{M} \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$$

with left and right Leibniz rule and flip

$$\mathcal{M} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\sigma} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$$

Right Leibniz rule:

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi) f$$

σ 'brings' d to the left; in general $\sigma^2 \neq 1$

The de Rham σ necessarily of the form

$$\sigma(\xi \otimes \eta) = \eta \otimes \xi$$

The flip is necessarily $\mathcal{A}$-bilinear

Bimodule $\mathcal{A}$-connection: the couple (D, σ)

Linear connection: $\mathcal{M} = \Omega^1(\mathcal{A})$

We define the torsion map: $\Theta : \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A})$

by $\Theta = d - \pi \circ D$; it is left-linear and

$$\Theta(\xi)f - \Theta(\xi f) = \pi \circ (1 + \sigma)(\xi \otimes df)$$

We impose $\pi \circ (\sigma + 1) = 0$

Using σ one can also construct an extension

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{M} \xrightarrow{D_2} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{M}$$

by $D_2(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12} \circ (\xi \otimes D\eta)$

Metric

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{g} \mathcal{A}, \qquad g \circ \sigma \propto g$$

The linear connection is metric compatible if

$$g_{23} \circ D_2 = d \circ g$$

Example with differential calculus Ω^*_η over M_3^+:

$$\Omega^1_\eta \otimes_{M_3^+} \Omega^1_\eta = M_3^+$$

Therefore $\sigma = \mathrm{diag}(\mu, \mu - 1), \;\; \mu \in \mathbb{C}$

Define

$$\eta = \eta_1 - \eta_1^*, \quad \eta_{ij} = \eta_i \otimes \eta_j^*, \quad \zeta = \eta_1^* \otimes \eta_1$$

Then $\;\sigma(\eta_{ij}) = \mu\eta_{ij}, \qquad \sigma(\zeta) = -1$

The unique bilinear metric is given by

$$g(\eta_{ij}) = \eta_i\eta_j^* \in M_2, \qquad g(\zeta) = -e \in M_1$$

It is real on η_{ij} and imaginary on ζ

The unique covariant derivative is given by

$$D\xi = -\eta \otimes \xi + \sigma(\xi \otimes \eta)$$

The torsion vanishes

Thw connection is metric compatible if $\mu = 1$

The parallelizable case: $\Omega^1(\mathcal{A})$ is free as a left or right $\mathcal{A}$-module and has a special basis θ^a with

$$[f, \theta^a] = 0, \qquad 1 \leq a \leq n$$

and dual to a set of derivations $e_a = \operatorname{ad} \lambda_a$:

$$df = e_a f \theta^a = [\lambda_a, f]\theta^a = -[\theta, f], \qquad \theta = -\lambda_a \theta^a$$

As a bimodule 'Dirac operator' θ generates $\Omega^1(\mathcal{A})$

In terms of the basis:

$$D\theta^a = -\omega^a{}_{bc}\theta^b \otimes \theta^c, \qquad \omega^a{}_{bc} \in \mathcal{A}$$

$$\sigma(\theta^a \otimes \theta^b) = S^{ab}{}_{cd}\theta^c \otimes \theta^d, \quad S^{ab}{}_{cd} \in \mathcal{Z}(\mathcal{A})$$

Linear connection $\quad D\theta^a = -\theta \otimes \theta^a + \sigma(\theta^a \otimes \theta)$

A (more-or-less unique) metric: $\quad g(\theta^a \otimes \theta^b) = g^{ab}$

The connection is metric-compatible if

$$\omega^a{}_{bd} g^{de} + \omega^e{}_{fg} S^{af}{}_{bh} g^{hg} = 0$$

Consistency condition:

$$2\lambda_c\lambda_d P^{cd}{}_{ab} - \lambda_c F^c{}_{ab} - K_{ab} = 0$$

$P^{cd}{}_{ab}$ define the product in the algebra of forms:

$$\theta^a\theta^b = \pi(\theta^c \otimes \theta^d) = P^{ab}{}_{cd}\theta^c \otimes \theta^d$$

$F^c{}_{ab}$ are related to the 2-form $d\theta^a$:

$$d\theta^a = -\frac{1}{2}(F^a{}_{bc} - 2\lambda_e P^{(ae)}{}_{bc})\theta^b\theta^c$$

K_{ab} are related to the curvature of the 'Dirac operator':

$$d\theta + \theta^2 = \frac{1}{2}K_{ab}\theta^a\theta^b$$

The coefficients lie all in $\mathcal{Z}(\mathcal{A})$ $(\equiv \mathbb{C})$

When

$$P^{ac}{}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c)$$

the $F^c{}_{ab}$ are hermitian, K_{ab} anti-hermitian

Reality conditions on d:

$$(df)^* = df^*, \qquad (e_a f^*)^* = e_a f, \qquad \lambda_a^* = -\lambda_a$$

For general $f \in \mathcal{A}$ and $\xi \in \Omega^1(\mathcal{A})$ one has

$$(f\xi)^* = \xi^* f^*, \qquad (\xi f)^* = f^* \xi^*$$

There are $I^{ab}{}_{cd} \in \mathcal{Z}(\mathcal{A})$ such that

$$(\theta^a \theta^b)^* = \imath(\theta^a \theta^b) = I^{ab}{}_{cd} \theta^c \theta^d$$

and $J^{ab}{}_{cd} \in \mathcal{Z}(\mathcal{A})$ such that

$$(\theta^a \otimes \theta^b)^* = \jmath_2(\theta^a \otimes \theta^b) = J^{ab}{}_{cd} \theta^c \otimes \theta^d$$

Compatibility with the product: $\pi \circ \jmath_2 = \imath \circ \pi$

Therefore: $(\xi\eta)^* = -\eta^* \xi^*$

One finds also the relations

$$(f\xi\eta)^* = (\xi\eta)^* f^*, \qquad (f\xi \otimes \eta)^* = (\xi \otimes \eta)^* f^*$$

Reality conditions on D:

$$D\xi^* = (D\xi)^*, \qquad (\omega^a{}_{bc})^* = \omega^a{}_{de}(J^{de}{}_{bc})^*$$

From the Leibniz rules and the equalities

$$(D(f\xi))^* = D((f\xi)^*) = D(\xi^* f^*)$$

for all f one finds the conditions

$$(fD\xi)^* = (D\xi^*)f^*, \qquad (\xi \otimes \eta)^* = \sigma(\eta^* \otimes \xi^*)$$

The reality condition for the metric becomes

$$g((\xi \otimes \eta)^*) = (g(\xi \otimes \eta))^*, \qquad S^{ab}{}_{cd} g^{cd} = (g^{ba})^*$$

Define the curvature as the map

$$\text{Curv} : \Omega^1(\mathcal{A}) \longrightarrow \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

given by $\quad \text{Curv} = D^2 = \pi_{12} \circ D_2 \circ D$

Reality conditions on the curvature:

$$\mathrm{Curv}(\xi^*) = (\mathrm{Curv}(\xi))^*$$

We shall impose a stronger condition

$$D_2(\xi \otimes \eta)^* = (D_2(\xi \otimes \eta))^*$$

There are $J^{abc}{}_{def} \in \mathcal{Z}(\mathcal{A})$ such that

$$(\theta^a \otimes \theta^b \otimes \theta^c)^* = J^{abc}{}_{def}\theta^d \otimes \theta^e \otimes \theta^f$$

We find that

$$J^{abc}{}_{def} = J^{ab}{}_{pq}J^{pc}{}_{dr}J^{qr}{}_{ef} = J^{bc}{}_{pq}J^{aq}{}_{rf}J^{rp}{}_{de}$$

The second equality is the Yang-Baxter Equation

It becomes the braid equation for the map σ:

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}$$

6. Infinite-dimensional models

The example $\mathcal{A} = \mathcal{C}(V) \otimes M_n$ (Kaluza-Klein):

$$\Omega^1(M_n) \simeq \bigoplus_1^d M_n, \quad n >> d$$

Differential calculus: $\Omega^*(\mathcal{A}) = \Omega^*(V) \otimes \Omega^*(M_n)$

Therefore $\Omega^1(\mathcal{A}) = \Omega^1_h \oplus \Omega^1_v$ with

$$\Omega^1_h = \Omega^1(V) \otimes M_n, \qquad \Omega^1_v = \mathcal{C}(V) \otimes \Omega^1(M_n)$$

The differential df of $f \in \mathcal{A}$ is given by

$$df = d_h f + d_v f, \quad \theta^i = (\theta^\alpha, \theta^a)$$

Gauge group (left): $\mathcal{U}_n = \mathcal{C}(V) \otimes U_n$

We have $\Omega\psi = D^2\psi$ where (with $K_{ab} = 0$)

$$\Omega = \frac{1}{2}\Omega_{ij}\theta^i\theta^j = \frac{1}{2}F_{\alpha\beta}\theta^\alpha\theta^\beta + D_\alpha\phi_b\theta^\alpha\theta^b + \frac{1}{2}\Omega_{ab}\theta^a\theta^b$$

with $\quad \Omega_{ab} = [\phi_a, \phi_b] - C^c{}_{ab}\phi_c$

The electromagnetic action for (A, ϕ) is

$$S[A,\phi] = \frac{1}{4}\mathrm{Tr}\int F_{\alpha\beta}F^{\alpha\beta}$$

$$+ \frac{1}{2}\mathrm{Tr}\int D_\alpha\phi_a D^\alpha\phi^a - \int V(\phi)$$

with $\quad V(\phi) = -\frac{1}{4}\mathrm{Tr}\,(\Omega_{ab}\Omega^{ab})$

If $d = 3$ the (hidden) 'quantum cell' has area

$$2\pi\hbar \simeq \frac{1}{n}4\pi r^2$$

The potential $V(\phi)$ vanishes when ϕ lies on a gauge orbit of a representation of SU_2

There are $\quad p(n) \simeq \dfrac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \quad$ such orbits

The gravitational action is Einstein-Hilbert in 'dimension' $4 + d$ (plus Gauss-Bonnet terms)

The examples $\mathbb{C}_q^n$ and $\mathbb{R}_q^n$:

The $SO_q(n)$ braid matrix has a decomposition

$$\hat{R} = qP_s - q^{-1}P_a + q^{1-n}P_t$$

with the P_s, P_a, P_t mutually orthogonal and

$$P_s + P_a + P_t = 1$$

For example $\quad {P_t}^{ij}{}_{kl} = (g^{mn}g_{mn})^{-1}g^{ij}g_{kl} \quad$ and

$$\begin{aligned} g_{il}\,\hat{R}^{\pm 1 lh}{}_{jk} &= \hat{R}^{\mp 1 hl}{}_{ij}\,g_{lk}, \\ g^{il}\,\hat{R}^{\pm 1 jk}{}_{lh} &= \hat{R}^{\mp 1 ij}{}_{hl}\,g^{lk} \end{aligned}$$

The g_{ij} is the q-deformed euclidean metric

The q-euclidean 'spaces' $\mathbb{C}_q^n$: generators x^i with

$${P_a}^{ij}{}_{kl}x^kx^l = 0$$

Real q-euclidean 'spaces' $\mathbb{R}_q^n$: $q \in \mathbb{R}^+$ and

$$(x^i)^* = x^j g_{ji}$$

The 'length' squared $\quad r^2 = g_{ij}x^ix^j = (x^i)^*x^i$
generates the center of $\mathbb{R}_q^n$

Extend $\mathbb{R}_q^n$ by r, r^{-1} and the dilatator Λ

$$x^i \Lambda = q \Lambda x^i, \qquad \Lambda^* = \Lambda^{-1}$$

Center now trivial; set $d\Lambda = 0$

Two $SO_q(n)$-covariant differential calculi:

$$x^i \xi^j = q \hat{R}^{ij}{}_{kl} \xi^k x^l$$

for $\Omega^1(\mathbb{R}_q^n)$ and

$$x^i \bar{\xi}^j = q^{-1} \hat{R}^{-1ij}{}_{kl} \bar{\xi}^k x^l$$

for $\bar{\Omega}^1(\mathbb{R}_q^n)$; no real calculus

Extend the involution to $\Omega^1(\mathcal{A}) \oplus \bar{\Omega}^1(\mathcal{A})$ by

$$(\xi^i)^* = \bar{\xi}^j g_{ji}$$

There exists a frame $(\theta^a, \bar{\theta}^a)$ with $(\theta^a)^* = \bar{\theta}^b g_{ba}$

The example $\mathbb{R}^1_q$:

The algebra $\mathbb{R}^1_q$: x and Λ with $\quad x\Lambda = q\Lambda x$

We choose x hermitian and $q \in (1, \infty)$

Write $x = q^y$: $\quad \Lambda^{-1} y \Lambda = y + 1$

Differential calculus $\Omega^*(\mathbb{R}^1_q)$:

$$xdx = qdxx, \qquad dx\Lambda = q\Lambda dx$$

Introduce $z = q^{-1}(q-1) > 0$ and choose

$$\lambda_1 = -z^{-1}\Lambda$$

The calculus is defined by $e_1 = \mathrm{ad}\,\lambda_1$

Adjoint derivation $e_1^\dagger$ of e_1: $\quad e_1^\dagger f = (e_1 f^*)^*$

Since Λ is unitary e_1 is not real; use $\bar{\Omega}^*(\mathbb{R}^1_q)$:

$$x\bar{d}x = q^{-1}\bar{d}xx, \qquad \bar{d}x\Lambda = q\Lambda\bar{d}x$$

based on $\bar{e}_1$ formed from $\bar{\lambda}_1 = -\lambda_1^*$: $\quad e_1^\dagger = \bar{e}_1$

The dual frames θ^1 and $\bar\theta^1$:

$$\theta^1 = \theta^1_1 dx, \quad \theta^1_1 = \Lambda^{-1} x^{-1},$$
$$\bar\theta^1 = \bar\theta^1_1 \bar d x, \quad \bar\theta^1_1 = q^{-1} \Lambda x^{-1}$$

Consider the element of $\mathbb{R}^1_q \times \mathbb{R}^1_q$:

$$\lambda_{R1} = (\lambda_1, \bar\lambda_1) = z^{-1}(-\Lambda, \Lambda^{-1})$$

The $e_{R1} = \mathrm{ad}\, \lambda_{R1}$ is real; the structure of

$$\Omega^*_R(\mathbb{R}^1_q) \subset \Omega^*(\mathbb{R}^1_q) \times \bar\Omega^*(\mathbb{R}^1_q)$$

is given by the relations $d_R \theta^1_R = 0, \quad (\theta^1_R)^2 = 0$

The forms θ^1, $\bar\theta^1$ and θ^1_R are exact

There are two torsion-free connections, one compatible with the unique local metric:

$$g(\theta^1_R \otimes \theta^1_R) = 1$$

The flip: $\sigma_R = 1$; the covariant derivative is real

Represent $\mathbb{R}^1_q$ on a Hilbert space $\mathcal{R}_q = \{|k\rangle\}$ by

$$x|k\rangle = q^k|k\rangle, \qquad \Lambda|k\rangle = |k+1\rangle$$

The element y has the representation

$$y|k\rangle = k|k\rangle$$

Extend to the differential calculus:

For the two elements dx and $\bar{d}x$:

$$dx|k\rangle = q^{k+1}|k+1\rangle, \qquad \bar{d}x|k\rangle = q^k|k-1\rangle$$

Then $\quad \theta^1 = 1, \quad \bar{\theta}^1 = 1$

The $d_R x$ can be represented by the operator

$$d_R x|k\rangle = q^k(q|k+1\rangle + \overline{|k-1\rangle})$$

We have placed a bar over the second copy of $\mathcal{R}_q$
On $\mathcal{R}_q \oplus \mathcal{R}_q$ we have the representation

$$\theta^1_R = 1$$

Interpretation of the metric in terms of observables since we have a representation of x and $d_R x$ on the Hilbert space $\mathcal{R}_q$

In this representation the distance s along the 'line' x is given by the expression

$$ds(k) = \|\sqrt{g'_{11}}d_R x(|k\rangle + \overline{|k\rangle})\| = \|\theta^1_R(|k\rangle + \overline{|k\rangle})\|$$

We have used here

$$g'^{11} = g(d_R x \otimes d_R x) = (e_{R1}x)^2 g(\theta^1_R \otimes \theta^1_R)$$

We find that

$$ds(k) = \|\,|k\rangle + \overline{|k\rangle}\,\| = 1$$

The 'space' is discrete and the spacing between 'points' is uniform

The example $\mathbb{R}_q^3$:

We set $x^a = (x^-, y, x^+), \quad h = \sqrt{q} - 1/\sqrt{q}$

The defining relations are

$$x^- y = q\, y x^-,$$

$$x^+ y = q^{-1} y x^+,$$

$$[x^+, x^-] = h y^2$$

The metric matrix is given by $g_{ij} = g^{ij}$ with

$$g_{ij} = \begin{pmatrix} 0 & 0 & 1/\sqrt{q} \\ 0 & 1 & 0 \\ \sqrt{q} & 0 & 0 \end{pmatrix}$$

By direct calculation one finds that

$$P_t{}^{ab}{}_{cd} \theta^c \theta^d = 0, \qquad P_s{}^{ab}{}_{cd} \theta^c \theta^d = 0$$

Therefore

$$P^{ab}{}_{cd} = P_{(a)}{}^{ab}{}_{cd}$$

Consider the elements $\lambda_a \in \mathbb{R}^3_q$ with

$$\lambda_- = +h^{-1} q \Lambda y^{-1} x^+,$$

$$\lambda_0 = -h^{-1} \sqrt{q} \Lambda y^{-1} r,$$

$$\lambda_+ = -h^{-1} \Lambda y^{-1} x^-$$

The $e_a = \mathrm{ad}\, \lambda_a$ are dual to the θ^a

Commutation relations identical to those of x^a:

$$\lambda_- \lambda_0 = q \lambda_0 \lambda_-,$$

$$\lambda_+ \lambda_0 = q^{-1} \lambda_0 \lambda_+,$$

$$[\lambda_+, \lambda_-] = h(\lambda_0)^2$$

These equations can be rewritten more compactly in the form

$$P^{ab}{}_{cd} \lambda_a \lambda_b = 0$$

This is the consistancy relation of the frame formalism with $C^a{}_{bc} = 0, \quad F_{ab} = 0$

Example: the Lobachevsky plane

Let $V = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{y} > 0\}$

A moving frame is given by

$$\theta^1 = \tilde{y}^{-1} d\tilde{x}, \;\; \theta^2 = \tilde{y}^{-1} d\tilde{y}, \quad ds^2 = \tilde{y}^{-2}(d\tilde{x}^2 + d\tilde{y}^2)$$

Introduce $\mathcal{A}_h$ with hermitian generators (x, y) and relation

$$[x, y] = -2ihy$$

A real frame is given by

$$\theta^1 = y^{-1} dx, \qquad \theta^2 = y^{-1} dy$$

The structure of $\Omega^*(\mathcal{A})$ is given by

$$(\theta^1)^2 = 0, \qquad (\theta^2)^2 = 0, \qquad \theta^1\theta^2 + \theta^2\theta^1 = 0$$

This algebra and differential calculus are invariant under the coaction of the Jordanian deformation of SL_2; Killing: $\underline{sl}(2, \mathbb{R})$

Metric: $g(\theta^a \otimes \theta^b) = g^{ab}$

The unique torsion-free, metric-compatible linear connection:

$$D\theta^1 = \theta^1 \otimes \theta^2, \qquad D\theta^2 = -\theta^1 \otimes \theta^1$$

The curvature map becomes

$$\mathrm{Curv}(\theta^1) = \theta^1\theta^2 \otimes \theta^2, \qquad \mathrm{Curv}(\theta^2) = -\theta^1\theta^2 \otimes \theta^1$$

Noncommutative Lobachevsky: $R_{1212} = -1$

Representation: introduce (ξ, η) with $[\xi, \eta] = 2ih$

Express: $x = \xi\eta - ih, \qquad y = \xi$

Find a representation of ξ and η

Define $\Lambda = e^{ix}, \quad q = e^{-2h}$

Then $y\Lambda = q\Lambda y$, which defines $\mathbb{R}^1_q$ with another differential calculus