Noncommutativity in type IIB superstring theory and T-duality

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BlagoFest, 12.-14. September 2011, Divčibare, Serbia

Outline of the talk

Superstrings

- Strings are object with one spatial dimension. During motion string sweeps a two dimensional surface called world sheet, parametrized by timelike parameter τ and spacelike one $\sigma \in [0\,,\pi]$. There are open and closed strings.
- Demanding presence of fermions in theory, we obtain superstring theory.
- There are five consistent superstring theories.
- Three approaches to superstring theory: NSR (Neveu-Schwarz-Ramond) (world sheet supersymmetry), GS (Green-Schwarz) (space-time supersymmetry) and pure spinor formalism (N. Berkovits, hep-th/0001035).

Superstring theories

1. Type I
   - Unoriented open and closed strings, $N = 1$ supersymmetry, gauge symmetry group $SO(32)$.
2. Type IIA
   - Closed oriented and open strings, $N = 2$ supersymmetry, nonchiral.
3. Type IIB
   - Closed oriented and open strings, $N = 2$ supersymmetry, chiral.
4. Two heterotic theories
   - Closed oriented strings, $N = 1$ supersymmetry, symmetry group either $SO(32)$ or $E_8 \times E_8$.

T-duality

- Changing of target space coordinates does not change the physical content of the theory.
- We perform Busher T-duality:
    - We notice that action has global shift symmetry along some directions.
    - Gauging - introducing of gauge fields $v_\pm$ and changing the corresponding drivatives $\partial_\pm \to \partial_\pm + v_\pm$ $(\sigma_\pm = \tau \pm \sigma)$.
    - On the equation of motion for $v_\pm$ we obtain the dual action.
    - Neumann boundary conditions transform to Dirichlet ones and vice versa (open strings).
- Bosonic and fermionic T-duality.

Type IIB superstring theory

- We use the action of type IIB superstring theory in pure spinor formulation up to the quadratic terms (without ghost terms)

$$
\begin{aligned}
S = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \\
+ \quad \int_\Sigma d^2\xi \left[-\pi_\alpha \partial_- (\theta^\alpha + \Psi^\alpha_\mu x^\mu) + \partial_+ (\bar\theta^\alpha + \bar\Psi^\alpha_\mu x^\mu) \bar\pi_\alpha + \frac{1}{2\kappa} \pi_\alpha F^{\alpha\beta} \bar\pi_\beta \right]
\end{aligned}
$$

- Definitions: $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$, Ψ^α_μ and $\bar\Psi^\alpha_\mu$ are NS-R fields and $F^{\alpha\beta}$ is R-R field strength. Momenta π_α and $\bar\pi_\alpha$ are canonically conjugated to θ^α and $\bar\theta^\alpha$. All spinors are Majorana-Weyl ones.
- All background fields are constant.

Hamiltonian

Hamiltonian is of the following form

$$H_c = \int d\sigma \mathcal{H}_c\,, \quad \mathcal{H}_c = T_- - T_+\,, \quad T_\pm = t_\pm - \tau_\pm\,,$$

where

$$\begin{aligned} t_\pm &= \mp\frac{1}{4\kappa} G^{\mu\nu} I_{\pm\mu} I_{\pm\nu}\,, \quad I_{\pm\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu} x'^\nu + \pi_\alpha \psi^\alpha_\mu - \bar\psi^\alpha_\mu \bar\pi_\alpha \\ \tau_+ &= (\theta'^\alpha + \psi^\alpha_\mu x'^\mu)\pi_\alpha - \frac{1}{4\kappa}\pi_\alpha F^{\alpha\beta}\bar\pi_\beta \\ \tau_- &= (\bar\theta'^\alpha + \bar\psi^\alpha_\mu x'^\mu)\bar\pi_\alpha + \frac{1}{4\kappa}\pi_\alpha F^{\alpha\beta}\bar\pi_\beta . \end{aligned}$$

Derivation of boundary condition

- As time translation generator Hamiltonian H_c must have well defined functional derivatives with respect to coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^\alpha)$ and their canonically conjugated momenta $(\pi_\mu, \pi_\alpha, \bar{\pi}_\alpha)$

$$\delta H_c = \delta H_c^{(R)} - \left[\gamma_\mu^{(0)} \delta x^\mu + \pi_\alpha \delta\theta^\alpha + \delta\bar{\theta}^\alpha \bar{\pi}_\alpha\right] \Big|_0^\pi .$$

- The first term is so called regular term. It does not contain τ and σ derivatives of coordinates and momenta variations.
- The second term has to be zero and we obtain boundary conditions. There are various choices of boundary conditions.

Boundary conditions as canonical constraints

- Let $\Lambda^{(0)}$ be a constraint. Then consistency of constraint demands that it is preserved in time

$$\Lambda^{(n)} \equiv \frac{d\Lambda^{(n-1)}}{d\tau} = \left\{ H_c \,, \Lambda^{(n-1)} \right\} \approx 0 \,. \quad (n = 1, 2, \dots)$$

- In all cases we will consider here, this is an infinite set of constraints. Using Taylor expansion we can rewrite this set of constraints in compact σ-dependent form

$$\Lambda(\sigma) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \Lambda^{(n)}(\sigma = 0) \,, \; \tilde{\Lambda}(\sigma) = \sum_{n=0}^{\infty} \frac{(\sigma - \pi)^n}{n!} \Lambda^{(n)}(\sigma = \pi) \,.$$

- Our goal - to find such boundary conditions which will make some of the T-dual background fields to be noncommutative parameters.

Busher bosonic T-duality

- Global shift symmetry exists $x^\mu \to x^\mu + a$.
- We introduce gauge fields $v_\pm^\mu$ and make change in the action $\partial_\pm x^\mu \to \partial_\pm x^\mu + v_\pm^\mu$.
- Additional term in the action

$$S_{gauge}(y, v_\pm) = \frac{1}{2}\kappa \int_\Sigma d^2\xi y_\mu(\partial_+ v_-^\mu - \partial_- v_+^\mu),$$

  where y_μ is Lagrange multiplier.
- After fixing x^μ to zero, on the equation of motion for $v_\pm^\mu$ we obtain dual action. The dual fields will be denoted as initial ones with one additional $\star$ - ${}^\star G^{\mu\nu}$, ${}^\star B^{\mu\nu}$, ${}^\star \Psi^{\mu\alpha}$, ${}^\star \bar\Psi^{\mu\alpha}$ and ${}^\star F^{\alpha\beta}$. The target space of the dual theory is spanned by y_μ.

Boundary conditions

- For coordinates x^μ we choose Neumann boundary conditions.
- In order to preserve $N = 1$ SUSY from initial $N = 2$, for fermionic coordinates we choose

$$(\theta^\alpha - \bar\theta^\alpha)\Big|_0^\pi = 0 \implies (\pi_\alpha - \bar\pi_\alpha)\Big|_0^\pi = 0\,.$$

- σ-dependent form of boundary conditions are the second class constraints. Instead Dirac brackets we will solve them.

Solution of boundary conditions

- Solving boundary conditions, we get

$$x^\mu(\sigma) = q^\mu - 2\Theta^{\mu\nu}\int_0^\sigma d\sigma_1 p_\nu + \frac{\Theta^{\mu\alpha}}{2}\int_0^\sigma d\sigma_1(p_\alpha + \bar{p}_\alpha)\,,$$

$$\theta^\alpha(\sigma) = \eta^\alpha - \Theta^{\mu\alpha}\int_0^\sigma d\sigma_1 p_\mu - \frac{\Theta^{\alpha\beta}}{4}\int_0^\sigma d\sigma_1(p_\beta + \bar{p}_\beta)\,,$$

$$\bar{\theta}^\alpha(\sigma) = \bar{\eta}^\alpha - \Theta^{\mu\alpha}\int_0^\sigma d\sigma_1 p_\mu - \frac{\Theta^{\alpha\beta}}{4}\int_0^\sigma d\sigma_1(p_\beta + \bar{p}_\beta)\,,$$

where

$$\eta^\alpha \equiv \frac{1}{2}(\theta^\alpha + \Omega\bar{\theta}^\alpha)\,,\quad \bar{\eta}^\alpha \equiv \frac{1}{2}(\Omega\theta^\alpha + \bar{\theta}^\alpha)\,,$$

$$p_\alpha \equiv \pi_\alpha + \Omega\bar{\pi}_\alpha\,,\quad \bar{p}_\alpha \equiv \Omega\pi_\alpha + \bar{\pi}_\alpha\,.$$

Noncommutativity

- Using the solution, we have

$$\{x^\mu(\sigma), x^\nu(\bar\sigma)\} = 2\Theta^{\mu\nu}\Delta(\sigma+\bar\sigma),$$

$$\{x^\mu(\sigma), \theta^\alpha(\bar\sigma)\} = -\Theta^{\mu\alpha}\Delta(\sigma+\bar\sigma),$$

$$\{\theta^\alpha(\sigma), \bar\theta^\beta(\bar\sigma)\} = \frac{1}{2}\Theta^{\alpha\beta}\Delta(\sigma+\bar\sigma).$$

Relation between T-duality and noncommutativity

- Up to the constants, noncommutativity parameters are Ω odd ($\Omega : \sigma \to \sigma$) combinations of T-dual background fields

$$\Theta^{\mu\nu} = \frac{1}{\kappa} {}^\star B^{\mu\nu} , \quad \Theta^{\mu\alpha} = \frac{1}{2\kappa} \left({}^\star\Psi^{\mu\alpha} + {}^\star\bar{\Psi}^{\mu\alpha} \right) , \quad \Theta^{\alpha\beta} = \frac{1}{2\kappa} {}^\star F_s^{\alpha\beta} .$$

- Noncommutativity parameters are $N = 1$ supermultiplet. Initial theory on the solution of boundary conditions (effective theory) has background fields which are also $N = 1$ supermultiplet. These two $N = 1$ supermultiplets are background fields of T-dual theory ($N = 2$ supermultiplet).

Basic facts

- In last years it was seen that tree level superstring theories on certain supersymmetric backgrounds admit a symmetry which is called fermionic T-duality.
- This is a redefinition of the fermionic worldsheet fields similar to the redefinition we perform on bosonic variables when we do an ordinary T-duality.
- Technically, the procedure is the same as in the bosonic case up to the fact that dualization will be done along θ^α and $\bar{\theta}^\alpha$ directions.

Fermionic T-dual fields

- Applying the same procedure we obtain

$$^{\star}G_{\mu\nu} = G_{\mu\nu} + 2\left[(\bar{\Psi}F^{-1}\Psi)_{\mu\nu} + (\bar{\Psi}F^{-1}\Psi)_{\nu\mu}\right] ,$$

$$^{\star}B_{\mu\nu} = B_{\mu\nu} + \left[(\bar{\Psi}F^{-1}\Psi)_{\mu\nu} - (\bar{\Psi}F^{-1}\Psi)_{\nu\mu}\right] ,$$

$$^{\star}\Psi_{\alpha\mu} = 4(F^{-1}\Psi)_{\alpha\mu} , \quad ^{\star}\bar{\Psi}_{\mu\alpha} = -4(\bar{\Psi}F^{-1})_{\mu\alpha} ,$$

$$^{\star}F_{\alpha\beta} = 16(F^{-1})_{\alpha\beta} .$$

- Two successive dualizations give the initial background fields as in the case of bosonic T-duality.

Choice of boundary conditions

- Reminder-boundary term:

$$\left[\gamma_\mu^{(0)}\delta x^\mu + \pi_\alpha \delta\theta^\alpha + \delta\bar{\theta}^\alpha \bar{\pi}_\alpha\right]\Big|_0^\pi = 0\,.$$

- Now we choose Dirichlet boundary conditions for all coordinates

$$\dot{x}^\mu\Big|_0^\pi = 0\,, \quad \dot{\theta}^\alpha\Big|_0^\pi = 0\,, \quad \dot{\bar{\theta}}^\alpha\Big|_0^\pi = 0\,.$$

- σ-dependent boundary conditions are second class constraints.

Solution of boundary conditions

- The solution for coordinates is trivial, while for momenta we have

$$x^\mu(\sigma) = \tilde{q}^\mu(\sigma)\,, \pi_\mu = \tilde{p}_\mu - 2\kappa\,{}^\star B_{\mu\nu}\tilde{q}'^\nu + \frac{\kappa}{2}\left({}^\star\bar{\Psi}_{\alpha\mu}\theta_a'^\alpha + \bar{\theta}_a'^\alpha\,{}^\star\Psi_{\alpha\mu}\right)\,,$$
$$\theta^\alpha(\sigma) = \theta_a^\alpha(\sigma)\,, \quad \pi_\alpha = \tilde{p}_\alpha - \frac{\kappa}{8}\bar{\theta}_a'^\beta\,{}^\star F_{\beta\alpha} + \frac{\kappa}{2}\,{}^\star\bar{\Psi}_{\alpha\mu}\tilde{q}'^\mu\,,$$
$$\bar{\theta}^\alpha(\sigma) = \bar{\theta}_a^\alpha(\sigma)\,, \quad \bar{\pi}_\alpha = \tilde{\bar{p}}_\alpha - \frac{\kappa}{8}\,{}^\star F_{\alpha\beta}\theta_a'^\beta - \frac{\kappa}{2}\,{}^\star\Psi_{\alpha\mu}\tilde{q}'^\mu\,,$$

- Coordinates are commutative.

Momenta noncommutativity

- $$\{P_\mu(\sigma), P_\nu(\bar\sigma)\} = \Theta_{\mu\nu}\Delta(\sigma+\bar\sigma)\,, \{P_\mu(\sigma), P_\alpha(\bar\sigma)\} = \bar\Theta_{\mu\alpha}\Delta(\sigma+\bar\sigma)\,,$$
$$\left\{P_\mu(\sigma), \bar P_\alpha(\bar\sigma)\right\} = \Theta_{\alpha\mu}\Delta(\sigma+\bar\sigma)\,, \left\{P_\alpha(\sigma), \bar P_\beta(\bar\sigma)\right\} = \Theta_{\alpha\beta}\Delta(\sigma+\bar\sigma)\,,$$
$$\{P_\alpha(\sigma), P_\beta(\bar\sigma)\} = \left\{\bar P_\alpha(\sigma), \bar P_\beta(\bar\sigma)\right\} = 0\,,$$
- Here we have
$$\Theta_{\mu\nu} = 2\kappa\,{}^\star B_{\mu\nu}\,, \bar\Theta_{\mu\alpha} = \frac{\kappa}{2}\,{}^\star\bar\Psi_{\mu\alpha}$$
$$\Theta_{\alpha\mu} = -\frac{\kappa}{2}\,{}^\star\Psi_{\alpha\mu}\,, \Theta_{\alpha\beta} = -\frac{\kappa}{8}\,{}^\star F_{\beta\alpha}\,.$$

Concluding remarks

- We considered relation between noncommutativity and T-duality in type IIB superstring theory.
- Boundary conditions are treated as canonical constraints. Applying canonical approach we solve boundary conditions and express initial variables in terms of the effective ones.
- If we choose Neumann boundary conditions for x^μ and $N = 1$ supersymmetric conditions for fermionic variables, we show that coordinate noncommutativity parameters are (up to some constants) bosonic T-dual fields. Momenta are commutative.
- If we choose Dirichlet boundary conditions for all variables, we obtain momenta noncommutativity with fermionic T-dual fields as noncommutativity parameters.