# Poincare Gauge Theory of Gravity

## Gravity: New ideas for unsolved problems

In honour of 67th birthday of Milutin Blagojević
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Dragan S. Popović

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# Poincaré Gauge Theory of Gravitation and its Hamiltonian Formulation (*).

M. Blagojević

*Department of Theoretical Physics, Boris Kidrič Institute*
*P.O. Box* 522, 11001 *Belgrade, Yugoslavia*

I. A. Nikolić and D. S. Popović

*Institute of Physics - P.O. Box* 57, *Belgrade, Yugoslavia*

Dj. Živanović

*Department of Physics, Faculty of Natural Sciences and Mathematics*
*P.O. Box* 550, 11001 *Belgrade, Yugoslavia*
*Department of Theoretical Physics, Boris Kidrič Institute*
*P.O. Box* 522, 11001 *Belgrade, Yugoslavia*

(ricevuto il 15 Settembre 1980)

**Summary.** — Poincaré gauge approach to the theory of gravitation is formulated. It has a very close resemblance to the usual procedure for gauging internal symmetries. By using Dirac's systematic method for systems with constraints, Einstein-Cartan form of Poincaré gauge theory is put into Hamiltonian form, by means of a time gauge and by treating tetrad and connection coefficients as independent variables.

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# GRAVITATIONAL SINGULARITY IN POINCARÉ GAUGE THEORY ☆

M. BLAGOJEVIĆ
*Department of Theoretical Physics, B. Kidrič Institute, Belgrade, Yugoslavia*

D.S. POPOVIĆ
*Institute of Physics, 11001 Belgrade, Yugoslavia*

and

Dj. ŽIVANOVIĆ
*Department of Physics, Faculty of Natural Sciences and Mathematics, Belgrade, Yugoslavia*

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The problem of the existence of a gravitational singularity in Poincaré gauge theory is studied. It is found that, under certain conditions, a singularity can be avoided in the case of homogeneous and isotropic space.

One of the reasons for studying quantum gravity is the hope that it might resolve the problem of the spacetime singularities of classical general relativity (GR) [1]. However, there exists no satisfying formulation of quantum gravity today. Another possible resolution of the problem might be in the change of the classical theory itself. An indication of the direction in which GR could be changed lies in the attractive idea of unification of all interactions, based on a gauge principle. Following this line of thought, we present here the investigation of the problem of spacetime singularities in Poincaré gauge theory of gravity, whose structure is, by now, pretty well understood [2–4].

The most general gravitational action, which can be obtained by gauging the Poincaré group, is of the form

$$A = \int d^4x\, b(\mathcal{L}_m + \mathcal{L}_g), \qquad (1)$$

where $\mathcal{L}_m$ and $\mathcal{L}_g$ are matter and gravitational lagrangian densities, respectively, and $b$ is the determinant of the inverse tetrad coefficients. Demanding that the gravitational lagrangian density be at most quadratic in gauge field strengths and invariant under space reflection, one obtains [3]

$$\mathcal{L}_g = aF + \mathcal{L}_T + \mathcal{L}_F, \qquad (2)$$

where $\mathcal{L}_T$ contains three terms quadratic in the irreducible components of the translation field strength-torsion $T_{ijk}$,

$$\mathcal{L}_T = \alpha t_{ijk} t^{ijk} + \beta v_i v^i + \gamma a_i a^i,$$

$\mathcal{L}_F$ contains six terms (only five of them being independent, due to the Bach–Lanzosh identity [3]) quadratic in the irreducible components of the Lorentz field strength-curvature $F_{ijkl}$,

$$\mathcal{L}_F = a_1 A_{ijkl} A^{ijkl} + a_2 B_{ijkl} B^{ijkl} + a_3 C_{ijkl} C^{ijkl} + a_4 E_{ij} E^{ij} + a_5 I_{ij} I^{ij} + a_6 F^2,$$

$F$ is the linear invariant formed from $F_{ijkl}$ and $a$ is a constant. The existence of nine constants makes this theory rather complicated. In what follows, we first investigate the question of the existence of repulsive gravitational forces, which could be important when discussing the equation of state [5]. Then we study the problem of gravitational singularity within homogeneous and isotropic space. The limitations on the values of the constants, stemming from the particle content of the theory, will be used.

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*Repulsive gravitational forces.* In order to study the particle content of this theory of gravity, we use the weak field approximation of the field equations based on (1) and (2). Beside the massless graviton, there exist six irreducible components of torsion (tordions) with $J^P = 2^{\pm}, 1^{\pm}, 0^{\pm}$, obeying equations of the Klein–Gordon type. We consider here a choice of constants such that all tordions are massive and $a = -1/16\pi G$ ($G$ is Newton's gravitational constant) [3].

The exchange of massless gravitons of spin 2 generates an *attractive* force between two static objects. The same thing happens in the case of any even-spin particle exchange, whereas odd-spin particle exchange produces a *repulsive* force [6]. This conclusion depends on the structure of source current. It is interesting to find out whether Poincaré gauge theory of gravity, based on the Lagrangian density (2) and a spin one-half matter field, can produce a repulsive force between static objects by the exchange of $J^P = 1^+, 1^-$ tordions. This might give us some indirect insight into the singularity behaviour of the theory.

The Klein–Gordon type equations of motion for the vector $\widetilde{V}_i$ ($J^P = 1^-$) and the axial vector $\widetilde{A}_i$ ($J^P = 1^+$) tordions, following from action (1), are given in ref. [3]. One can show that the solution for $\widetilde{V}_i$, in static approximation, vanishes outside the source position, and that the solution for $\widetilde{A}_i$ is of the form

$$\widetilde{A}_0 = 0,$$

$$\widetilde{A}_a \sim [s_a - (s \cdot \nabla / m_{\mathrm{A}}^2)\partial_a]\, r^{-1} \exp(-m_{\mathrm{A}} r) \quad (a = 1, 2, 3), \tag{3}$$

where $s$ is the spin of the source and $m_{\mathrm{A}}$ is the mass parameter. The repulsive force between macroscopic objects with average spin zero is absent. The existence or nonexistence of repulsive forces is spin dependent and a weak field effect. It might have an influence on the singularity behaviour of the theory via the form of the equation of state [5].

*Homogeneous and isotropic space.* We now turn our attention to the investigation of spacetime singularities in the case of homogeneous and isotropic space with metric

$$ds^2 = dt^2 - R^2(t) \times [(1 - kr^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)] \quad (4)$$

and torsion

$$T^1{}_{10} = T^2{}_{20} = T^3{}_{30} \equiv T(t) \quad (5)$$

the other components being zero [7,8]. We also assume that the spin tensor of the matter field is zero, which seems to be a quite realistic assumption for describing *macroscopic* matter. By combining gravitational field equations, one obtains the generalized Friedmann cosmological equation [7] for scaling factor $R(t)$,

$$6k(b + \tfrac{2}{3}fF)^2 + 6[\dot{R}(b + \tfrac{2}{3}fF) + \tfrac{1}{3}f\dot{F}R]^2 + [\rho R^2 + \tfrac{1}{3}fF^2R^2 + 9\beta(k + \dot{R}^2)](b + \tfrac{2}{3}fF) = 0, \quad (6)$$

$$F = (\rho - 3p)/2b + (9\beta/b)(k + \dot{R}^2 + R\ddot{R})/R^2, \quad (7)$$

where $p$ denotes the pressure, $\rho$ is energy density of matter, dot denotes time derivative, $b = a - \frac{3}{2}\beta$ and $4f = a_5 + 12a_6$. As a consequence of the symmetry requirements (4) and (5), there are only two constants in the general action (2) that survive: $f$ and $\beta$.

Since for $f = 0$, eq. (6) simplifies to the singular GR cosmological equation, we will assume hereafter that $f \neq 0$. If $\beta \neq 0$, than there is only one propagating tordion with $J^P = 0^+$. The condition of positive-definite energy and positive mass squared then leads to $f > 0$ (eq. (4.6) in the fourth reference in ref. [3]). If $\beta = 0$ none of tordions propagate and the sign of $f$ is arbitrary.

The conservation law

$$\dot{\rho} + 3(\dot{R}/R)(\rho + p) = 0, \quad (8)$$

which is a consequence of the gravitational field equations, and the equation of state $p = \varphi(\rho)$, in conjunction with the generalized Friedmann equation (6), determine the scaling factor $R(t)$. We will try to find out whether there are solutions $R(t)$ which are regular at the point $t = 0$, where GR gives singular behaviour for any $k = +1, 0, -1$ [9]; more precisely, we will try to see *whether $R(t)$ can have a minimum at $t = 0$, with a large-scale behaviour similar to the standard one in GR.*

Let us consider separately two cases: $\beta = 0$ and $\beta \neq 0$.

(a) $\beta = 0$, $f$ either positive or negative. In this case the cosmological equation (6) takes the simpler form

$$6k(a + \tfrac{2}{3}fF)^2 + 6[\dot{R}(a + \tfrac{2}{3}fF) + \tfrac{1}{3}f\dot{F}R]^2 + (\rho + \tfrac{1}{3}fF^2)(a + \tfrac{2}{3}fF)R^2 = 0, \qquad (9)$$

with $F = (\rho - 3p)/2a$. Using the arguments of ref. [7] one can show that for $fF > 0$ there is an upper limit for the energy density $\rho$, defined by the equation

$$a + \tfrac{2}{3}fF < 0.$$

Due to the conservation law (8), there is a corresponding lower limit of $R(t)$. It is related to the *minimum* of the curve $R(t)$ at the point $t = 0$, where GR shows singular behaviour [9]. If $f > 0$, then $fF > 0$ reduces to $F > 0$, i.e. $p > \frac{1}{3}\rho$. Such a condition can in principle be realized by taking into account repulsion between nucleons at very small distances [6]. Besides nuclear forces, repulsion could be produced also by gravitation itself, as discussed above. Although possible in principle, an equation of state consistent with $p > \frac{1}{3}\rho$ is related to such high matter densities, that its reality is far from being established.

The statement about the existence of nonsingular solution for $R(t)$ in this case should be understood in the light of this remark. If, on the other hand, we consider the case $f < 0$, then a singularity can be avoided by systems for which $p < \frac{1}{3}\rho$ [7]. Although there are no propagating tordions, the theory is essentially different from **GR**.

For $p = \frac{1}{3}\rho$ (radiation) eq. (9) reduces to the standard GR form.

(b) $\beta \neq 0$, $f$ positive. To find out the relevant properties of $R(t)$ is now much more difficult than in the previous case.

In some neighbourhood of a regular point $T$, the solution of eq. (6) can be represented in the following form

$$R(t) = \sum_{k=0}^{\infty} R_k (t - T)^k, \qquad (10)$$

with $R_k = R_k(T)$. Expecting to have a minimum for $T = 0$, we will put $R_1(T = 0) = 0$. Using the invariance of eq. (6) under time reversal, we also put $R_{2n+1}(T = 0) = 0$, $n = 1, 2, \ldots$ This condition, however, does not guarantee that we are really investigating the point $T = 0$, it is related to all extremal points, with respect to which the solution is (locally) symmetric. To isolate the point $T = 0$ from the other symmetric extremal points we will use the fact that extrema which reproduce the GR condition for a maximum (with $k = +1$) in the limit $f, \beta \to 0$, are not related to the point $T = 0$.

We will consider separately two interesting cases of dusty matter ($p = 0$) and radiation ($p = \frac{1}{3}\rho$).

(i) $p = 0$. Using the conservation law $\rho R^3 = D =$ const. [9] and the expansion (10) with $R_{2n+1} = 0$ in eq. (6), we obtain a set of recurrence relations for $R_{2n}$. The first of these relations takes the form

$$(b + \tfrac{2}{3}fF_0)[6k(b + \tfrac{2}{3}fF_0) + D/R_0 + \tfrac{1}{3}fF_0^2R_0^2 + 9\beta k] = 0, \qquad (11)$$

where $F_0$ is the zeroth order term in expansion of $F$, eq. (7), in powers of $(t - T)$. When $f \to 0$, the equality to zero of the expression in square brackets gives the GR condition for an extremum. The interpretation that this extremum is a maximum is consistent with the other recurrence relations. Taking the relation between $R_0$ and $R_2$ to be the same as in GR,

$$4R_0R_2 + k = 0, \qquad (12)$$

one obtains an equation for $R_0$ and $k$ which is practically the same as in GR if

$$(fk/a)(1 - fk/b) \ll 1. \qquad (13)$$

The vanishing of the first factor in eq. (11) results in the condition

$$bR_0^2 + \tfrac{2}{3}f[D/2bR_0 + (9\beta/b)(2R_0R_2 + k)] = 0. \qquad (14)$$

By detailed inspection of this equation, which is of third order in $R_0$, one can see that it has positive solutions for $R_0$, with $R_2 > 0$, if a certain relation among $f, \beta, D$ and $k$ is satisfied. This means that *there exists a minimum* $R_0 > 0$ of $R(t)$ at the point $t = 0$, which is absent if $f, \beta \to 0$. By comparing the solution with the physical observations, one should find out whether this minimum is *physically* acceptable.

(ii) $p = \frac{1}{3}\rho$. The conservation law in this case is of the form $\rho R^4 = D =$ const. [9].

The first recurrence relation for the coefficients $R_{2n}$ is given by

$$(b + \tfrac{2}{3}fF_0)[6k(b + \tfrac{2}{3}fF_0) + D'/R_0 + \tfrac{1}{3}fF_0^2R_0^2 + 9\beta k] = 0. \qquad (15)$$

Vanishing of the expression in square brackets gives an equation which, in the limit $f, \beta \to 0$, reproduces the GR condition for a maximum (with $k = +1$). If one demands the relation among $R_0$, $R_2$ and $k$ to be the same as in GR,

$$2R_0R_2 + k = 0, \tag{16}$$

one finds that the GR condition for a maximum is exactly reproduced, irrespective of the values of $f$ and $\beta$. The vanishing of the first factor in eq. (15) gives

$$bR_0^2 + (6f\beta/b)(2R_0R_2 + k) = 0. \tag{17}$$

This quadratic equation has a positive solution for $R_0$, with $R_2 > 0$, if $f$, $\beta$ and $k$ satisfy a certain relation. This again means the *existence of a minimum* of $R(t)$ at $t = 0$.

In all the cases considered, it may happen that there is more then one solution for $R_0$. Then, matching with given initial conditions will decide which solution will be realized in a given case.

*Conclusion.* Starting with the most general gravitational lagrangian density in Poincaré gauge theory (which is at most quadratic in gauge field strengths and invariant under space reflection) we investigated the problem of the existence of space time singularities in the case of homogeneous and isotropic space.

In the special case of the Einstein–Cartan theory (with a lagrangian which is linear in the scalar curvature $F$) this problem has been studied in ref. [10]. Their description of spinning matter is, however, not compatible with the cosmological principle [8]. Furthermore, the general theory considered here does not reduce to GR if we assume that the spin tensor of the matter field is zero (reflecting the fact that the average spin of macroscopic objects is zero), as is the case in the Einstein–Cartan theory [2].

When $\beta = 0$, i.e. there are no propagating tordions, we found that there exists a minimum of the metric scaling function $R(t)$ at $t = 0$ if (i) $f > 0$ and $p > \frac{1}{3}\rho$, or (ii) $f < 0$ and $p < \frac{1}{3}\rho$. The equation of state, corresponding to the first case, is possible in principle but far from being really established. The second case is related to the more usual systems.

In the general case $\beta \neq 0$, corresponding to the existence of one scalar propagating tordion, we investigated local properties of $R(t)$ near $t = 0$, for dusty matter ($p = 0$) and radiation ($p = \frac{1}{3}\rho$). It is found, in both cases, that a minimum of $R(t)$ at $t = 0$ exists if the constants in the theory ($f, \beta, D$ or $D'$, $k$) satisfy a certain relation. It is possible, at the same time, to have large-scale behaviour of $R(t)$ closely resembling the standard GR case. The complete physical significance of the solutions of this type can be understood after carefull comparison with the corresponding observational data.