D-BRANES FROM CLASSICAL MACROSCOPIC STRINGS

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I. MEMBRANE DYNAMICS

Let us start with the brief recapitulation of the known results. The p-brane world-sheet equations are obtained from the covariant conservation equations of fundamental matter currents: nonsymmetric stress-energy $\tau^{\mu}{}_{\nu}$, and spin tensor $\sigma^{\lambda}{}_{\mu\nu}$. In the coordinate basis, these equations have the form [16]:

$$\left(D_\nu + \mathcal{T}^{\lambda}{}_{\nu\lambda}\right) \tau^{\nu}{}_{\mu} = \tau^{\nu}{}_{\rho}\mathcal{T}^{\rho}{}_{\mu\nu} + \frac{1}{2}\sigma^{\nu\rho\sigma}\mathcal{R}_{\rho\sigma\mu\nu}, \tag{1a}$$

$$\left(D_\nu + \mathcal{T}^{\lambda}{}_{\nu\lambda}\right) \sigma^{\nu}{}_{\rho\sigma} = \tau_{\rho\sigma} - \tau_{\sigma\rho}. \tag{1b}$$

Here, D_ν is the covariant derivative with nonsymmetric connection, $\mathcal{T}^{\lambda}{}_{\mu\nu}$ is torsion and $\mathcal{R}^{\mu}{}_{\nu\rho\sigma}$ stands for curvature. The covariant derivative D_ν is assumed to satisfy the metricity condition. It has been shown in [5] that the conservation equations (1) can be put in the form

$$\nabla_\nu \left(\theta^{\mu\nu} - \mathcal{D}^{\mu\nu}\right) = \frac{1}{2}\sigma_{\nu\rho\lambda}\nabla^{\mu}K^{\rho\lambda\nu}, \tag{2}$$

where $\theta^{\mu\nu}$ stands for the generalized Belinfante tensor

$$\theta^{\mu\nu} \equiv \tau^{(\mu\nu)} - \nabla_\rho\sigma^{(\mu\nu)\rho} - \frac{1}{2}K_{\lambda\rho}{}^{(\mu}\sigma^{\nu)\rho\lambda}.$$

This way, the conservation equations are rewritten in terms of the Riemannian covariant derivative ∇_μ, and the contortion $K^\lambda{}_{\mu\nu} \equiv -\frac{1}{2}\left(\mathcal{T}^\lambda{}_{\mu\nu} - \mathcal{T}_\nu{}^\lambda{}_\mu + \mathcal{T}_{\mu\nu}{}^\lambda\right)$. The notation

$$\mathcal{D}^{\mu\nu} \equiv K^{[\mu}{}_{\lambda\rho}\sigma^{\rho\lambda\nu]} + \frac{1}{2}K_{\lambda\rho}{}^{[\mu}\sigma^{\nu]\rho\lambda}$$

is introduced for convenience. In the particle case, this form of the conservation equations has been used in [17].

In this paper, we are interested in infinitely thin branes, and therefore, we restrict our analysis to the lowest approximation in the multipole expansion of [3, 4]:

$$\theta^{\mu\nu} = \int_{\mathcal{M}} d^{p+1}\xi\sqrt{-\gamma}\, T^{\mu\nu}\frac{\delta^{(D)}(x-z)}{\sqrt{-g}}, \tag{3a}$$

$$\sigma^{\lambda\mu\nu} = \int_{\mathcal{M}} d^{p+1}\xi\sqrt{-\gamma}\, S^{\lambda\mu\nu}\frac{\delta^{(D)}(x-z)}{\sqrt{-g}}. \tag{3b}$$

The surface $\mathcal{M}$ is defined by the equation $x^\mu = z^\mu(\xi)$ where ξ^a are the surface coordinates, and $T^{\mu\nu}(\xi)$ and $S^{\lambda\mu\nu}(\xi)$ are free coefficients. The induced metric is defined by $\gamma_{ab} \equiv g_{\mu\nu}u^\mu_a u^\nu_b$ and $u^\mu_a \equiv \partial z^\mu/\partial\xi^a$. In [4, 5], the decomposition (3) has been used as an ansatz for solving the conservation equations (2).

In this paper, we are interested in membranes ($p = 2$) characterized by maximally symmetric distribution of spin. It has already been shown in [6] that such membranes must have axial spin tensor of the form

$$S^{\lambda\mu\nu} = s\, e^{abc}u^\lambda_a u^\mu_b u^\nu_c\,, \tag{4}$$

where e^{abc} is the covariant Levi-Civita symbol, and s is a constant. This leads us to restrict our considerations to backgrounds with *totally antisymmetric torsion.* The computations are straightforward, and have already been done in [6]. They lead to the world-sheet equations

$$\nabla_a \left(m^{ab} u_{b\mu} \right) = \frac{s}{2} u^{\nu\lambda\rho} K_{\mu\nu\lambda\rho} \tag{5a}$$

and boundary conditions

$$n_a \left(m^{ab} u_{b\mu} + \frac{3s}{2} e^{abc} K_{bc\mu} \right) \Big|_{\partial\mathcal{M}} = 0 \,. \tag{5b}$$

Here, n^a is the unit boundary normal, and ∇_a stands for the total covariant derivative that acts on both types of indices [3–5]. The antisymmetric tensor $K_{\mu\nu\lambda\rho}$ is defined as

$$K_{\mu\nu\lambda\rho} \equiv \partial_\mu K_{\nu\lambda\rho} - \partial_\nu K_{\lambda\rho\mu} + \partial_\lambda K_{\rho\mu\nu} - \partial_\rho K_{\mu\nu\lambda} \,,$$

while $u^{\mu\nu\rho} \equiv e^{abc} u_a^\mu u_b^\nu u_c^\rho$, and $K_{ab\rho} \equiv u_a^\mu u_b^\nu K_{\mu\nu\rho}$ are introduced for convenience. The coefficients s and m^{ab} are the residual free parameters of the theory.

II. STRING DYNAMICS

In this section, we shall review the uncompromised results of [7], where dimensional reduction of a cylindrical membrane has been considered.

The arena for our considerations is a $(D+1)$-dimensional spacetime with one small compact dimension. It is parametrized by the coordinates X^M $(M = 0, 1, \ldots, D)$, which we divide into the "observable" coordinates x^μ $(\mu = 0, 1, \ldots, D-1)$, and the extra periodic coordinate y. In the limit of small extra dimension, we use the Kaluca-Klein ansatz

$$\partial_y K_{MNL} = 0, \qquad \partial_y G_{MN} = 0 \tag{6}$$

to model the contortion and metric. For the metric G_{MN}, we shall use the standard decomposition

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + \phi\, a_\mu a_\nu & \phi\, a_\mu \\ \phi\, a_\nu & \phi \end{pmatrix}, \tag{7}$$

while the *totally antisymmetric contortion* K_{MNL} is decomposed as

$$K_{MNL} = \Big(K_{\mu\nu\lambda}\,,\; K_{\mu\nu y} \Big),$$

with

$$
\begin{aligned}
K_{\mu\nu y} &\equiv k_{\mu\nu}\,, \\
K_{\mu\nu\lambda} &\equiv \mathcal{K}_{\mu\nu\lambda} + k_{\mu\nu}a_\lambda + k_{\nu\lambda}a_\mu + k_{\lambda\mu}a_\nu\,.
\end{aligned}
\tag{8}
$$

With respect to the residual symmetry transformations, the variables $\mathcal{K}_{\mu\nu\lambda}$ and $k_{\mu\nu}$, as well as $g_{\mu\nu}$ and ϕ, transform as tensors, while a_μ transforms as a connection.

Now, we consider a membrane wrapped around the extra compact dimension y. Its world-sheet $X^M = Z^M(\xi^A)$ is denoted by $\mathcal{M}_3$, and is chosen in the form

$$
x^\mu = z^\mu(\xi^a), \qquad y = \xi^2, \tag{9}
$$

where the world-sheet coordinates ξ^A $(A = 0,1,2)$ are divided into ξ^a $(a = 0,1)$ and ξ^2. This ansatz reduces the membrane world-sheet tangent vectors $U_A^M \equiv \partial Z^M/\partial\xi^A$ to $U_a^\mu = u_a^\mu \equiv \partial z^\mu/\partial\xi^a$, $U_2^\mu = U_a^y = 0$, and $U_2^y = 1$. In what follows, we shall refer to $x^\mu = z^\mu(\xi^a)$ as the string world-sheet, and denote it by $\mathcal{M}_2$.

In this section, the dimensional reduction is applied to the membrane equations (5) of section I. As a first step, the word-sheet equations (5) are rewritten using the $(D+1)$-dimensional notation:

$$
\nabla_A \left(m^{AB} U_{BM} \right) = \frac{s}{2} U^{NLR} K_{MNLR}\,, \tag{10a}
$$

$$
N_A \left(m^{AB} U_{BM} + \frac{3s}{2} e^{ABC} K_{BCM} \right) \Big|_{\partial\mathcal{M}_3} = 0\,. \tag{10b}
$$

The dimensional reduction of (10) with $m^{AB} = T\mathit{\Gamma}^{AB}$, where $\mathit{\Gamma}_{AB} \equiv G_{MN} U_A^M U_B^N$ stands for the induced membrane metric, has been studied in [6]. In [7], on the other hand, a more general stress-energy tensor has been considered:

$$m^{AB} = T\mathit{\Gamma}^{AB} + \mu^{AB}.$$

To violate the maximal symmetry of the membrane stress-energy $m^{AB} = T\mathit{\Gamma}^{AB}$ in a way which will preserve the maximal symmetry of the effective string after dimensional reduction, the additional constraint $\mu^{ab} = 0$ has been imposed. Thus,

$$\mu^{AB} = \begin{pmatrix} 0 & \jmath^a \\ \jmath^b & \omega - 2\jmath^c a_c \end{pmatrix}. \tag{11}$$

This specific decomposition ensures tensorial character of the new parameters $\jmath^a(\xi)$ and $\omega(\xi)$. We shall see later that $\jmath^a$ and ω are related to the electric and dilatonic charges of the effective string.

The dimensional reduction of the membrane equations (10) has successfully been done in [7]. As it turns out, the effective string equations are nicely simplified when expressed in terms of the rescaled quantities

$$\tilde{g}_{\mu\nu} \equiv \sqrt{\phi}\, g_{\mu\nu}\,, \qquad \tilde{\jmath}^a \equiv \phi \jmath^a\,.$$

Indeed, in terms of $\tilde{g}_{\mu\nu}$ and $\tilde{\jmath}^a$, the effective string dynamics has the simple form:

$$\tilde{\nabla}_a \tilde{\jmath}^a = 0\,, \tag{12a}$$

$$T\tilde{\nabla}_a u^a_\mu = \frac{3s}{2}\tilde{u}^{\nu\lambda}k_{\mu\nu\lambda} - f_{\mu\nu}\tilde{\jmath}^\nu + \frac{\omega}{2}\partial_\mu\phi\,. \tag{12b}$$

The totally antisymmetric tensor $k_{\mu\nu\lambda}$ is defined as

$$k_{\mu\nu\lambda} \equiv \partial_\mu k_{\nu\lambda} + \partial_\nu k_{\lambda\mu} + \partial_\lambda k_{\mu\nu}\,,$$

while $f_{\mu\nu} \equiv \partial_\nu a_\mu - \partial_\mu a_\nu$ and $\jmath^\mu \equiv u^\mu_a \jmath^a$. Similarly, the dimensional reduction of the boundary conditions (10b) yields

$$\tilde{n}_a\tilde{\jmath}^a\big|_{\partial\mathcal{M}_2} = 0\,, \tag{13a}$$

$$\tilde{n}_a\left(Tu^a_\mu + 3s\,\tilde{e}^{ab}u^\nu_b k_{\mu\nu}\right)\big|_{\partial\mathcal{M}_2} = 0\,. \tag{13b}$$

To summarize, the effective string dynamics is governed by the world-sheet equations (12), and boundary conditions (13). The constant parameters T and s define maximally symmetric distribution of stress-energy and spin, but arbitrary $\tilde{\jmath}^a(\xi)$ and $\omega(\xi)$ violate the string uniformity. While electrically neutral strings ($\tilde{\jmath}^a = 0$) can easily be made uniform by imposing the condition $\omega = \text{const.}$, this is not the case with electrically charged strings ($\tilde{\jmath}^a \neq 0$). In what follows, we shall separately analyze these two cases.

III. ELECTRICALLY NEUTRAL STRING

In this section, we shall restrict our considerations to strings with maximally symmetric distribution of matter. In the simplest scenario, we simply get rid of the isotropy violating current $\tilde{\jmath}^a$ by putting it to zero:

$$\tilde{\jmath}^a = 0$$

With this, we are left with three free coefficients to parametrize our string equations. They define an electrically neutral string with constant tension and spin, and arbitrary dilatonic charge. In the maximally symmetric case, the charge ω is constrained to have a constant value, but we shall keep it arbitrary in the forthcoming analysis.

The string dynamics is obtained by substituting $\tilde{\jmath}^a = 0$ into (12) and (13). This way, we obtain the world-sheet equations

$$T\tilde{\nabla}_a u^a_\mu = \frac{3s}{2}\tilde{u}^{\nu\lambda}k_{\mu\nu\lambda} + \frac{\omega}{2}\partial_\mu\phi\,, \tag{14a}$$

and boundary conditions

$$\tilde{n}_a\left(Tu^a_\mu + 3s\,\tilde{e}^{ab}u^\nu_b k_{\mu\nu}\right)\Big|_{\partial\mathcal{M}_2} = 0\,. \tag{14b}$$

As expected, no coupling to the electromagnetic field is present. An interesting feature of the obtained dynamics is that the world-sheet projection of the equation (14a) does not identically vanish. Instead, it gives the constraint $\omega\, u^\mu_a \partial_\mu \phi = 0$, which reduces to

$$\partial_a \phi = 0 \tag{15}$$

in the generic case $\omega \neq 0$. The condition (15) tells us that the string world-sheet is embedded in a surface of constant ϕ. Precisely,

> *electrically neutral string with nontrivial dilatonic charge is constrained to live in one of the surfaces $\phi(x) = const.$*

The choice $\omega = 0$, on the other hand, brings us back to the trivial case $\mu^{AB} = 0$, already considered in [6]. It is characterized by the absence of dilaton coupling.

Let us now construct an action functional for our equations (14). It is immediately seen that this can not be done without auxiliary fields. Indeed, the string equations that follow from a reparametrization invariant action must be orthogonal to the world-sheet, which is not the case with our (14). To solve this problem, we introduce the auxiliary electromagnetic field $A_a(\xi)$ that exclusively lives on the string world-sheet. The needed action functional is then searched for in the form

$$I = T \int d^2\xi \sqrt{-\gamma} \Big[1 + \frac{1}{2}\, e^{ab} \left(B_{ab} + \Phi F_{ab} \right) \Big] , \tag{16}$$

where $\gamma_{ab} \equiv G_{\mu\nu}(x) u^\mu_a u^\nu_b$, $B_{ab} \equiv B_{\mu\nu}(x) u^\mu_a u^\nu_b$ and $F_{ab} \equiv \partial_b A_a - \partial_a A_b$. The external fields $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$ and $\Phi(x)$ are referred to as the target space metric, antisymmetric field and dilaton, respectively. The action (16) is varied with respect to the independent variables $x^\mu(\xi)$ and $A_a(\xi)$. As a result, the world-sheet equations and boundary conditions

$$\nabla_a u^a_\mu = \frac{1}{2} u^{\nu\lambda} B_{\mu\nu\lambda} + \frac{F}{2}\,\partial_\mu \Phi\,, \tag{17a}$$

$$n_a \left(u^a_\mu + e^{ab} u^\nu_b B_{\mu\nu} \right) \big|_{\partial\mathcal{M}_2} = 0\,, \tag{17b}$$

and the constraint

$$\partial_a \Phi = 0 \tag{17c}$$

are obtained. Here, $B_{\mu\nu\lambda} \equiv \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$, and $F \equiv e^{ab} F_{ab}$. As we can see, the auxiliary variable A_a remains undetermined, and therefore, the quantity $F(\xi)$ can be considered a free coefficient of the theory. With this, the equations (17) take the form of our (14) and (15). Indeed, the identification of external fields

$$G_{\mu\nu} = \tilde{g}_{\mu\nu}\,, \qquad B_{\mu\nu} = \frac{3s}{T}\,k_{\mu\nu}\,, \qquad \Phi = \phi\,, \tag{18}$$

and the free coefficients $F = \omega/T$ establishes the $1-1$ correspondence between the two sets of equations.

The action functional (16) is not the unique action that governs the dynamics of electrically neutral string with nontrivial dilatonic charge. An attractive possibility is to introduce the auxiliary metric $h_{ab}(\xi)$ along with the auxiliary electromagnetic field $A_a(\xi)$. This can be done in a variety ways, but we choose the action

$$\begin{aligned} I = \ & T \int d^2\xi \sqrt{-h} \Big[G_{\mu\nu}(x) u^\mu_a u^\nu_b h^{ab} \\ & + B_{\mu\nu}(x) u^\mu_a u^\nu_b e^{ab} + \Phi(x) \left(F + R^{(2)} \right) \Big], \end{aligned} \tag{19}$$

as it is closest to the string σ-model considered in string theory literature [10–15]. The $2d$ curvature $R^{(2)}$ is constructed out of the auxiliary metric h_{ab}, and $F \equiv e^{ab} F_{ab}$ stands for the field strength of the auxiliary electromagnetic field A_a. It is an exclusive feature of $2d$ spacetimes to allow scalars linear in the electromagnetic field strength.

The action (19) is varied with respect to the independent variables $x^\mu(\xi)$, $h_{ab}(\xi)$ and $A_a(\xi)$. As a result, the *equations that coincide with (17) are obtained*. It is only that the role of the free coefficient F in (17) is now played by the new free coefficient $\Omega \equiv F + R^{(2)} - \nabla^2 \chi$. The coefficient $\Omega(\xi)$ is a remnant of the auxiliary variables h_{ab} and A_a. In particular, χ stands for the undetermined conformal factor of the auxiliary metric h_{ab}.

The preceding considerations have demonstrated that the action functional (19) correctly governs the dynamics of electrically neutral effective string in Riemann-Cartan spacetime. Moreover, this action is seen to share many features with the string σ-model action of [10–15]. In fact, the two actions differ in one instance only: the auxiliary electromagnetic field strength present in our action is missing in the string σ-model. As a consequence, the dilaton couplings of the two models differ by the fact that *our string is additionally constrained to live in a surface* $\phi = const$. On the other hand, the result of [6] that macroscopic strings couple to metric and torsion the same way as fundamental strings couple to the low-energy symmetric and antisymmetric string fields is still valid. It has been derived in [6] in the absence of dilaton, but we have demonstrated here that it holds true in any case.

Finally, it should be noted that our effective string is a macroscopic classical object, which has a-priori nothing to do with the fundamental string of the string theory. The background fields $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$ and $\Phi(x)$ in (19) are just components of the dimensionally reduced spacetime metric and torsion, as shown in (18).

IV. ELECTRICALLY CHARGED STRING

We start with the observation that the equation (12b) implies the constraint $\omega \tilde{\jmath}^\mu \partial_\mu \phi = 0$, which reduces to

$$\tilde{\jmath}^a \partial_a \phi = 0 \tag{20}$$

in the generic case $\omega \neq 0$. The general solution of this constraint has the form

$$\tilde{\jmath}^a = e\, e^{ab} \partial_b \phi \,, \tag{21}$$

where $e(\xi)$ is the residual coefficient that defines the distribution of electric charge along the string. The isotropy violating vector valued coefficient $\tilde{\jmath}^a(\xi)$ is thereby replaced by the scalar $e(\xi)$. In what follows, *we shall restrict to the maximally symmetric case* $e = const$.

Before we continue, let us note that the general solution (21) is incompatible with the constraint $\partial_a \phi = 0$. Indeed, if $\partial_a \phi = 0$, the equation (20) is identically satisfied for any value of the current $\tilde{\jmath}^a$, while the general solution (21) implies $\tilde{\jmath}^a = 0$. To avoid this inconsistency, *we are led to adopt* $\partial_a \phi \neq 0$ in the subsequent considerations.

With these restrictions, the conservation equations (12a) are trivially satisfied, and the world-sheet equations (12b) are simplified by using the redefined contortion

$$\tilde{k}_{\mu\nu} \equiv k_{\mu\nu} - \frac{e}{3s} \phi f_{\mu\nu} \,,$$

and redefined dilatonic charge

$$\tilde{\omega} \equiv \omega + e u^{\mu\nu} f_{\mu\nu} \,.$$

Indeed, in terms of $\tilde{k}_{\mu\nu}$ and $\tilde{\omega}$, the world-sheet equations (12b) are rewritten as

$$T \tilde{\nabla}_a u^a_\mu = \frac{3s}{2} \tilde{u}^{\nu\lambda} \tilde{k}_{\mu\nu\lambda} + \frac{\tilde{\omega}}{2} \partial_\mu \phi \,.$$

As we can see, the resultant world-sheet equations contain no coupling to the electromagnetic field. In fact, we can prove that they do not contain the dilaton coupling, either. To see this, we project the above equation to the world-sheet, and obtain the familiar constraint

$$\tilde{\omega} \partial_a \phi = 0 \,.$$

This time, however, the generic choice $\tilde{\omega} \neq 0$ is forbidden by the fact that the whole analysis of this section rests upon the assumption $\partial_a \phi \neq 0$. Indeed, if the dilatonic charge $\tilde{\omega}$ was assumed to have a non-zero value, the resulting constraint $\partial_a \phi = 0$ would contradict our introductory assumptions. Therefore, we are led to constrain $\tilde{\omega}$ to zero,

$$\tilde{\omega} = 0$$

With this, the world-sheet equations are reduced to

$$T\tilde{\nabla}_a u^a_\mu = \frac{3s}{2}\tilde{u}^{\nu\lambda}\tilde{k}_{\mu\nu\lambda} \tag{22}$$

showing that our macroscopic string can not probe the background electromagnetic and dilaton fields. As opposed to the electrically neutral case of the preceding section, the electrically charged string is not constrained to live on a surface of constant ϕ. In fact, it is only the string interior that has this freedom, because the string boundary is additionally constrained by the boundary conditions (13). In terms of $\tilde{k}_{\mu\nu}$, these are rewritten as

$$e\, v^a \partial_a \phi\big|_{\partial\mathcal{M}_2} = 0\,, \tag{23a}$$

$$\tilde{n}_a \left[T u^a_\mu + \tilde{e}^{ab} u^\nu_b \left(3s\, \tilde{k}_{\mu\nu} + e\phi f_{\mu\nu}\right)\right]\Big|_{\partial\mathcal{M}_2} = 0\,, \tag{23b}$$

where $v^a \equiv d\zeta^a/d\tau$ is the boundary tangent vector (the string boundary is defined by $\xi^a = \zeta^a(\tau)$). As we can see, the coupling to the electromagnetic field reappears in the boundary conditions, and the corresponding coupling constant is defined as the boundary value of $e\phi$. That it is indeed constant is seen from the boundary condition (23a), which is rewritten as

$$\frac{d\phi}{d\tau} = 0 \tag{24}$$

in the generic case $e \neq 0$. Thus, the value of the dilaton field is constant along the string boundary, which implies that the string ends are confined to live on surfaces of constant ϕ.

Precisely,

> *each end of an electrically charged string lives in a surface $\phi(x) = const.$*

At the same time, its interior freely moves in all D dimensions. This way, a *classical analogue of the known D-brane concept is obtained.* In the generic situation, the surfaces of constant ϕ define codimension-1 branes. Higher codimension branes, on the other hand, are not so easy to define. This is because the higher codimension surfaces $\phi(x) = const$ violate the condition $\partial_\mu \phi \neq 0$ needed for the boundary condition (23a) to make sense. What one can do is to consider specific dilaton configurations characterized by a dense distribution of regular codimension-1 surfaces near a singular lower-dimensional brane. A particularly interesting situation arises in brane-world scenarios where the bulk dilaton is localized in all but four dimensions. In such spacetimes, the string ends are attached to 3-branes.

To summarize, we have shown that the effective string dynamics is governed by the world-sheet equations (22), and boundary conditions (23). The equations are parametrized by three constant parameters. The constant parameters T and s define uniform distribution of stress-energy and spin along the string, while parameter e stems from the conserved electric charge located at the string ends. As a consequence, only the string boundary couples to the external electromagnetic field.

It is seen that the two string ends may have different coupling constants. Indeed, if the string is stretched between two different surfaces, let us say

$$\phi(x) = \phi_1 \qquad \text{and} \qquad \phi(x) = \phi_2 \,,$$

the respective coupling constants $e\phi_1$ and $e\phi_2$ generally differ. For this reason, the *generic electromagnetic coupling can not be removed by a simple redefinition of external fields.* However, if we restrict our considerations to strings whose both ends are attached to a single brane, let us say $\phi(x) = \phi_0$ the electromagnetic coupling is removed by a simple redefinition

$$\tilde{k}_{\mu\nu} \to \tilde{k}_{\mu\nu} - \frac{e\phi_0}{3s} f_{\mu\nu}$$

The resultant string equations almost coincide with those of the trivial case $\tilde{\jmath}^a = \omega = 0$. The only difference is that the ends of such electrically charged string are additionally confined to live in a single surface $\phi(x) = \text{const}$.

At the end, let us construct the corresponding action functional, and compare our result with the string theory literature. It is immediately seen that our world-sheet equations (22) coincide with the corresponding string theory equations. However, the boundary conditions (23) are more restrictive than the Dirichlet boundary conditions employed in string theory. In the next section, we shall demonstrate how a small modification of the membrane constituent matter leads to the improved effective string dynamics. Precisely, we shall consider a membrane with massive boundary, which ultimately leads to an effective string with particles attached to its ends. The resulting effective string dynamics and classical D-branes will follow from an action functional that coincides with that of string theory.

V. MEMBRANE WITH MASSIVE BOUNDARY

In this section, we shall consider a membrane with massive, spinless boundary. To this end, we modify the stress-energy tensor (3a) as follows:

$$\theta^{\mu\nu} = \int_{\mathcal{M}_3} d^3\xi \sqrt{-\gamma}\, T^{\mu\nu} \frac{\delta^{(D)}(x - z(\xi))}{\sqrt{-g}} + \int_{\partial\mathcal{M}_3} d^2\lambda \sqrt{-\delta}\, t^{\mu\nu} \frac{\delta^{(D)}(x - z(\zeta))}{\sqrt{-g}} \,. \tag{25}$$

Here, $t^{\mu\nu}(\lambda)$ are free coefficients, $\delta_{ij} \equiv \gamma_{ab} v_i^a v_j^b$ is the boundary induced metric, and $v_i^a \equiv \partial\zeta^a/\partial\lambda_i$. The membrane boundary is defined by $\xi^a = \zeta^a(\lambda)$ and is denoted by $\partial\mathcal{M}_3$. The spinless character of the boundary matter is taken into account by keeping the spin tensor (3b) unmodified.

It is immediately seen that the modification (25) does not influence the membrane interior. For this reason, the corrections are expected only at the boundary, while the world-sheet equations retain the same form as derived in the previous sections. The procedure for solving the covariant conservation equations (2) is the same as in section I. We adopt the same assumptions about the maximal symmetry of the spin tensor, while keeping the stress-energy arbitrary. As a consequence, the membrane world-sheet equations remain unchanged, while boundary conditions acquire an additional term. Using the $(D+1)$-dimensional notation, the resulting membrane world-sheet equations are given by (10a), while boundary conditions become

$$\nabla_i \left(\rho^{ij} V_{jM} \right) = N_A \left(m^{AB} U_{BM} + \frac{3s}{2} e^{ABC} K_{BCM} \right) . \tag{26}$$

Here, $m^{AB}(\xi)$ and $\rho^{ij}(\lambda)$ are the residual free coefficients, and ∇_i is a total covariant derivative that acts on all three types of indexes [4, 5]. While m^{AB} stands for the stress-energy of the membrane interior, the ρ^{ij} represents the stress-energy of the membrane boundary.

What we are really interested in is the dynamics of effective string obtained after $D+1 \to D$ dimensional reduction. The procedure is the same as in section II, leading to the same world-sheet equations (12). In the case of electrically charged string, these reduce to (22), irrespectively of the presence of massive particles on its ends. In what follows, we shall examine their boundary conditions.

The dimensional reduction of boundary conditions goes as follows. The membrane boundary $\partial\mathcal{M}_3$ is given by $\xi^A = \zeta^A(\lambda^i)$, where λ^i $(i = 1, 2)$ are the boundary coordinates. It is chosen in the form

$$\xi^a = \zeta^a(\lambda^0)\,, \qquad \xi^2 = \lambda^1\,, \tag{27}$$

so that the boundary tangent vectors $V_i^A \equiv \partial\zeta^A/\partial\lambda^i$ become $V_0^a = v^a \equiv d\zeta^a/d\tau$, $V_1^a = V_0^2 = 0$, and $V_1^2 = 1$. The notation $\lambda^0 \equiv \tau$ is introduced for convenience. In what follows, $\xi^a = \zeta^a(\tau)$ will be referred to as the string boundary $\partial\mathcal{M}_2$. The induced metric on $\partial\mathcal{M}_2$ is defined by $ds^2 = v^2 d\tau^2$, where $v^2 \equiv \gamma_{ab} v^a v^b$.

Before we proceed, let us introduce a proper decomposition of the boundary stress-energy ρ^{ij}. Following the ideas of section II, we shall relax the maximal symmetry of $\rho^{ij} = m\delta^{ij}$ by adopting the decomposition

$$\rho^{ij} = m\delta^{ij} + \theta^{ij}. \tag{28a}$$

Here, m is a constant, and $\theta^{ij} = \theta^{ij}(\tau)$, in accordance with the adopted Kaluza-Klein ansatz. We shall see later that the constant m is related to the masses of the attached particles. The role of θ^{ij} is to violate the maximal symmetry of the boundary stress-energy $\rho^{ij} = m\delta^{ij}$ in a way which will preserve the maximal symmetry of the effective string boundary after dimensional reduction. To this end, we adopt the additional constraint $\theta^{00} = 0$, lading to

$$\theta^{ij} \equiv \begin{pmatrix} 0 & \pi \\ \pi & \theta - 2a\pi \end{pmatrix} \tag{28b}$$

This specific decomposition ensures tensorial character of the new parameters $\pi(\tau)$ and $\theta(\tau)$. With respect to reparametrizations $\tau' = \tau'(\tau)$, π transforms as a contravariant vector, and θ as a scalar. We shall see later that π and θ are related to the electric and dilatonic charges of the attached particles.

We can now dimensionally reduce the boundary conditions (26). Using the known form of the electric current (21), the y component takes the form

$$\pi = \frac{q - e\phi}{\phi^{5/4}\sqrt{-\tilde{v}^2}} \,,$$

showing that the role of π is taken over by the new constant q. While e defines uniform distribution of electric charge along the string, the constant q stands for the electric charge of the attached particle. This is why both string ends must have the same e, but allow different values of q. The μ components are obtained in a similar way. In terms of our previously redefined fields, they read

$$\tilde{m}\tilde{\nabla}\left(\frac{\tilde{v}_\mu}{\tilde{v}^2}\right) = \tilde{n}_a \left(T u^a_\mu + \tilde{e}^{ab} u^\nu_b Q_{\mu\nu} \right) + \tilde{\theta}\partial_\mu \phi \,, \tag{29}$$

where $\tilde{m} \equiv m\phi^{-1/4}$ and $\tilde{\theta} \equiv (\theta/2 + m/4\phi)\phi^{-1/4}$ are new constant parameters, while

$$Q_{\mu\nu} \equiv 3s\,\tilde{k}_{\mu\nu} + qf_{\mu\nu}$$

is introduced to shorten the expressions. The projection of (29) to the string boundary immediately yields the constraint $\tilde{\theta}v^\mu\partial_\mu\phi = 0$, which in the generic situation $\tilde{\theta} \neq 0$, reduces to

$$\frac{d\phi}{d\tau} = 0$$

Thus, *the string ends are confined to live in surfaces of constant ϕ irrespectively of the presence of massive particles.* The constant parameters T and s represent the string tension and spin, whereas $\tilde{m}$ and q stand for the conserved particle mass and electric charge. The remaining free coefficient $\tilde{\theta}$ is the only arbitrary function of τ in the string equations.

The action functional that governs the world-sheet equations (22), and boundary conditions (29) is constructed as follows. First, the existence of a surface that traps the string ends is taken for granted. It is promoted into an external constraint to be unconditionally satisfied without questioning its origin. This way, the variation of yet to be constructed action becomes subject to Dirichlet conditions. Second, we note that the only purpose of the undetermined coefficient $\tilde{\theta}$ in (29) is to tell us that the remaining part of (29) is orthogonal to the surface $\phi =$ const. Indeed, the constraint (29) can equivalently be written as

$$P_{\parallel\mu}{}^{\rho}\left[\tilde{m}\tilde{\nabla}\left(\frac{\tilde{v}_\rho}{\tilde{v}^2}\right) - \tilde{n}_a\left(Tu^a_\rho + \tilde{e}^{ab}u^\nu_b Q_{\rho\nu}\right)\right] = 0 \qquad (30)$$

where $P_{\parallel\mu}{}^{\rho}$ is the projector to the surface $\phi =$ const. As we can see, neither dilaton nor $\tilde{\theta}$ coefficient appear in this form of the equation (29).

Now, it is not difficult to check that the needed action functional has the form

$$I = \int d^2\xi \sqrt{-\tilde{\gamma}} \left(T + \frac{3s}{2} \tilde{u}^{\mu\nu} \tilde{k}_{\mu\nu} \right) + \tilde{m} \oint d\tilde{s} + q \oint a_\mu dx^\mu \quad (31)$$

The first integral in (31) goes over the string world-sheet $\mathcal{M}_2$, while the last two sweep the string boundary $\partial\mathcal{M}_2$. The induced metric on $\partial\mathcal{M}_2$ is defined by $d\tilde{s}^2 = \tilde{v}^2 d\tau^2$.

The *action functional (31) is varied respecting the Dirichlet boundary conditions* $\phi = const$. As a result, the world-sheet equations (22), and boundary conditions (30) (equivalently (29)) are obtained. The two coupling constants, $\tilde{m}$ and q, stand for the mass and electric charge of the particles attached to the string ends. In the limit $\tilde{m}, q \to 0$ (light neutral particles), the conventional string theory result is recovered. We can say that we indeed succeeded in obtaining the macroscopic D-brane analogue.

In the end, let us emphasize once more that our derivations have nothing to do with the conventional string theory. The action functional (31) describes a stringlike extended object in dimensionally reduced Riemann-Cartan spacetime, and nothing more. It is only the form of this action that brings the associations with string theory. Indeed, one can not help noticing that our macroscopic string couples to metric and torsion the same way as fundamental string couples to the low energy string fields. This may be just a coincidence, but we believe it deserves the attention of the scientific community.

VI. CONCLUDING REMARKS

In this paper, we have analyzed the behavior of a cylindrical membrane wrapped around the extra compact dimension of a $(D+1)$-dimensional Riemann-Cartan spacetime. The membrane constituent matter is specified in terms of its stress-energy and spin tensors. A membrane with maximally symmetric distribution of stress-energy and spin has already been considered in [6]. After dimensional reduction, such a membrane has been shown to reduce to a string that couples to the metric and torsion the same way as fundamental string couples to the low-energy string fields $G_{\mu\nu}$ and $B_{\mu\nu}$. In [7], the condition of maximal symmetry used in [6] has been relaxed. The effective string has been shown to carry two more charges, and to additionally couple to the electromagnetic and scalar fields of the dimensionally reduced geometry. Unfortunately, the precise form of these couplings has been missed, owing to the erroneous analysis in the final sections of [7].

In this paper, we have derived how exactly the effective string couples to the effective background geometry. In particular, we have discovered that coupling of an electrically charged string to the external electromagnetic field is located on the string boundary. The form of the coupling coincides with that found in string theory literature, but the string ends are additionally constrained to live on surfaces $\phi(x) = \text{const}$. This way, a macroscopic analogue of the known D-brane concept has been obtained. We have shown that the concept survives irrespectively of the presence of massive particles on the string ends. In the electrically neutral case, when the electromagnetic coupling is absent, we have demonstrated that the resulting string dynamics follows from the action functional which almost coincides with the string σ-model action of [10–15]. The two actions differ in one instance only: the auxiliary electromagnetic field present in our action is missing in the string σ-model. As a consequence, the dilaton couplings of the two models differ by the fact that our string is additionally constrained to live in a surface $\phi(x) = \text{const}$. On the other hand, the result of [6] that macroscopic strings couple to metric and torsion the same way as fundamental strings couple to the low-energy symmetric and antisymmetric string fields has been confirmed.

Let us say something about the dynamics of Riemann-Cartan geometry itself. We have already established correspondence between the macroscopic string dynamics in Riemann-Cartan spacetime, and the fundamental string dynamics in the low-energy string backgrounds. In this correspondence, the string background fields $G_{\mu\nu}$, $B_{\mu\nu}$, A_μ and Φ are related to the metric and torsion of the Riemann-Cartan spacetime. An interesting challenge would be to establish the correspondence on the level of background field equations. There have been attempts in literature to rewrite the low energy string field action in geometric terms [18, 19]. These have not been very successful though, as they included some unnatural constraints to be imposed on torsion prior to varying the action. If we stay with membranes, however, the construction of the needed action is quite simple. One should start with the low energy string field action, and replace the symmetric field with the spacetime metric, and the 3-form field with the axial component of the torsion. As an example, we may take the action of 11-dimensional supergravity as our starting point. It describes the low-energy limit of M-theory, and its bosonic part is a functional of the metric and the 3-form field. In the low-energy approximation, the suggested geometrization scheme leads to

$$\mathbf{I} = \int d^{11}X\sqrt{-G}\left(\mathbf{R} + K^{MNLR}K_{MNLR}\right),$$

where $\mathbf{R}$ stands for the 11-dimensional scalar curvature, and K_{MNLR} is the antisymmetrized derivative of the axial contortion K_{MNL}. This is a purely geometric action that governs the dynamics of 11-dimensional metric and torsion. After dimensional reduction, it turns into an action that governs the dynamics of 10-dimensional fields $g_{\mu\nu}$, a_μ, ϕ, $k_{\mu\nu}$ and $\mathcal{K}_{\mu\nu\lambda}$. It takes the form $I = I_1 + I_2$ where

$$I_1 = \int d^{10}x\sqrt{-\tilde{g}}\, e^{-\tilde{\phi}}\left(\tilde{R} + 4\tilde{k}^{\mu\nu\lambda}\tilde{k}_{\mu\nu\lambda} + \frac{5}{6}\tilde{\phi}^{,\mu}\tilde{\phi}_{,\mu}\right)$$
$$I_2 = \int d^{10}x\sqrt{-\tilde{g}}\left(-\frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mathcal{K}^{\mu\nu\lambda\rho}\mathcal{K}_{\mu\nu\lambda\rho}\right).$$

Here, the dimensionally reduced action has been rewritten in terms of the rescaled metric $\tilde{g}_{\mu\nu}$ and redefined torsion $\tilde{k}_{\mu\nu}$, which appear in the string world-sheet equations of the preceding sections. As for the redefined dilaton $\tilde{\phi} \equiv \ln \phi^{3/2}$, it does not explicitly appear in our world-sheet equations. Even so, the form of these equations is not significantly changed when expressed in terms of $\tilde{\phi}$. Indeed, the surfaces of constant ϕ are the same as those of constant $\tilde{\phi}$, while the other changes can be neutralized by a proper redefinition of the string free parameters. We can say that the fields of $I_1 + I_2$ basically coincide with those appearing in our string world-sheet equations.

Let us now compare our geometric action $I_1 + I_2$ with the similar low-energy string field actions appearing in literature. In fact, by the very construction, we already know that I_1+I_2 is a geometric counterpart of the action that describes the bosonic part of the low-energy sector in type IIA superstring theory. Indeed, the low-energy limit of type IIA superstring theory is known to coincide with dimensionally reduced 11-dimensional supergravity whose bosonic part we used as a starting point in constructing $I = I_1 + I_2$. Thus, we have established the correspondence between the low-energy string fields and Riemann-Cartan geometry. As it must be clear by now, this correspondence agrees with the correspondence (18) established on the level of string world-sheet equations.

Finally, let us say something about dynamics of the D-brane itself. As we have seen, the D-brane world-sheet is defined as a surface of constant ϕ. Obviously, the surface equations are closely related to the background field equations. As an example, we could study the background field equations of the geometric action $I_1 + I_2$. Such an attempt, however, generically leads to a non-local brane dynamics. A promising way to obtain a reasonable result would be to employ an external branelike matter source, with the idea to confine the ϕ field. If we could achieve this, the constant dilaton surfaces would be dense on the brane, and rare otherwise. This way, the massive branelike source would take the role of the classical D-brane of the preceding section. The investigation along these lines, however, lies outside the scope of this paper, and will be considered elsewhere.

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